Journal of Approximation Theory **108**, 119–160 (2001)
doi:10.1006/jath.2000.3490, available online at http://www.idealibrary.com on IDEAL®

Weierstrass' Theorem in Weighted Sobolev Spaces

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Communicated by Doron S. Lubinsky

Received February 10, 2000; accepted March 31, 2000;
published online January 18, 2001

We characterize the set of functions which can be approximated by polynomials with the following norm

$$\|f\|_{W^{k,\infty}([a,b],w)} := \sum_{j=0}^{k} \|f^{(j)}\|_{L^\infty([a,b],w_j)},$$

for a big class of weights $w_0, w_1, ..., w_k$

Key Words: Weierstrass' theorem; Sobolev spaces; weights.

1. INTRODUCTION

If I is any compact interval, Weierstrass' theorem says that $C(I)$ is the biggest set of functions which can be approximated by polynomials in the norm $L^\infty(I)$, if we identify, as usual, functions which are equal almost everywhere. There are many generalizations of this theorem (see, e.g., the monograph [L]).

Here we study the same problem with the norm $L^\infty(I, w)$ defined by

$$\|f\|_{L^\infty(I,w)} := \operatorname*{ess\,sup}_{x\in I} |f(x)|\, w(x), \tag{1.1}$$

where w is a weight, i.e., a non-negative measurable function, and we use the convention $0 \cdot \infty = 0$. Observe that (1.1) is not the usual definition of the L^∞ norm in the context of measure theory, although it is the correct one when we work with weights (see, e.g., [BO]). If $w = (w_0, ..., w_k)$ is a

[1] Research partially supported by a grant from DGES (MEC), Spain.

0021-9045/01 $35.00

vectorial weight, we also study this problem with the Sobolev norm $W^{k,\infty}(\Delta, w)$ defined by

$$\|f\|_{W^{k,\infty}(\Delta, w)} := \sum_{j=0}^{k} \|f^{(j)}\|_{L^\infty(\Delta, w_j)},$$

where $\Delta := \bigcup_{j=0}^{k} \operatorname{supp} w_j$. It is obvious that $W^{0,\infty}(\Delta, w) = L^\infty(\Delta, w)$.

Weighted Sobolev spaces is an interesting topic in many fields of mathematics (see, e.g., [HKM, K, Ku, KO, KS, and T]). In [ELW1], [EL], and [ELW2] the authors study some examples of Sobolev spaces for $p=2$ with respect to general measures instead of weights, in relation with ordinary differential equations and Sobolev orthogonal polynomials. The papers [RARP1], [RARP2], [R1], and [R2] are the beginning of a theory of Sobolev spaces with respect to general measures for $1 \leqslant p \leqslant \infty$. This theory plays an important role in the location of the zeroes of the Sobolev orthogonal polynomials (see [LP, RARP2, and R1]). The location of these zeroes allows us to prove results on the asymptotic behaviour of Sobolev orthogonal polynomials (see [LP]).

Now, let us state the main results here. We refer to the definitions in the next section. Throughout the paper, the results are numbered according to the section where they are proved.

We denote by $P^{k,\infty}(\Delta, w)$ $(k \geqslant 0)$ the set of functions which can be approximated by polynomials in the norm $W^{k,\infty}(\Delta, w)$, where we identify, as usual, functions which are equal almost everywhere. We must remark that the symbol $P^{k,\infty}(\Delta, w)$ has a slightly different meaning in [RARP1], [RARP2], [R1], and [R2].

First, we have results for the case $k=0$.

THEOREM 2.1. *Let us consider a closed interval I and a weight $w \in L^\infty_{loc}(I)$, such that the set S of singular points of w in I has zero Lebesgue measure. Then we have* $\overline{C^\infty(\mathbf{R}) \cap L^\infty(I, w)} = \overline{C(\mathbf{R}) \cap L^\infty(I, w)} = H$, *with*

$$H = \{ f \in C(I \setminus S) \cap L^\infty(I, w) \text{: for each } a \in S,$$
$$\exists l_a \in \mathbf{R} \text{ such that } \operatorname*{ess\,lim}_{x \in I,\, x \to a} |f(x) - l_a|\, w(x) = 0 \},$$

where the closures are taken in $L^\infty(I, w)$. If $a \in S$ is of type 1, *we can take as l_a any real number. If $a \in S$ is of type* 2, $l_a = \operatorname{ess\,lim}_{w(x) \geqslant \varepsilon,\, x \to a} f(x)$ *for $\varepsilon > 0$ small enough. Furthermore, if I is compact we also have* $P^{0,\infty}(I, w) = H$.

If $f \in H \cap L^1(I)$, I is compact, and S is countable, we can approximate f by polynomials with the norm $\|\cdot\|_{L^\infty(I, w)} + \|\cdot\|_{L^1(I)}$.

The following are two of the main results for $k \geqslant 1$.

THEOREM 5.3. *Let us consider a compact interval I and a vectorial weight $w=(w_0, ..., w_k) \in L^\infty(I)$ such that $w_k^{-1} \in L^1(I)$. Then we have*

$$P^{k,\infty}(I, w) = \{ f: I \to \mathbf{R} / f^{(k-1)} \in AC(I) \text{ and } f^{(k)} \in P^{0,\infty}(I, w_k) \}.$$

THEOREM 5.4. *Let us consider a compact interval I and a vectorial weight $w=(w_0, ..., w_k) \in L^\infty(I)$ such that the set of singular points for w_k in I has zero Lebesgue measure. Assume that there exist $a_0 \in I$, an integer $0 \leqslant r < k$, and constants $c, \delta > 0$ such that*

(1) $w_{j+1}(x) \leqslant c\,|x-a_0|\, w_j(x)$ *in* $[a_0-\delta, a_0+\delta] \cap I$, *for* $r \leqslant j < k$,

(2) $\int_{I \setminus [a_0-\varepsilon, a_0+\varepsilon]} w_k^{-1} < \infty$, *for every* $\varepsilon > 0$,

(3) *if* $r > 0$, a_0 *is* $(r-1)$*-regular.*

Then we have

$$\begin{aligned} P^{k,\infty}(I, w) = \{ f: I \to \mathbf{R} / f^{(k-1)} \in AC_{loc}(I \setminus \{a_0\}), f^{(k)} \in P^{0,\infty}(I, w_k), \\ \exists l \in \mathbf{R} \text{ with } \operatorname*{ess\,lim}_{x \in I,\, x \to a_0} |f^{(r)}(x) - l|\, w_r(x) = 0, \\ \operatorname*{ess\,lim}_{x \in I,\, x \to a_0} f^{(j)}(x)\, w_j(x) = 0, \\ \text{for } r \leqslant j < k \text{ if } r < k-1, \text{ and } f^{(r-1)} \in AC(I) \text{ if } r > 0 \}. \end{aligned}$$

This result gives the characterization of $P^{k,\infty}(I, w)$ for the case of Jacobi weights (see Corollary 5.1). There are other characterizations of $P^{k,\infty}(I, w)$ with weaker hypothesis on w_k (see Theorems 5.5, 5.6, and 5.7). There are also results for weights which can be obtained by "gluing" simpler ones (see Theorems 5.1 and 5.2).

We have results for non-bounded intervals (see Theorem 6.1 and Propositions 6.1, 6.2, and 6.3 in Section 6). These results deal with the case of Laguerre, Freud, and fast decreasing degree weights.

The analogue of Weierstrass' theorem with the norms $W^{k,p}(\Delta, \mu)$ (with $1 \leqslant p < \infty$ and μ a vectorial measure) can be founded in [RARP2] and [R2].

Now we present the notations we use.

Notations. Throughout the paper $k \geqslant 0$ denotes a fixed natural number. Also, all the weights are non-negative Borel measurable functions defined on a subset of $\mathbf{R}$; if a weight is defined in a proper subset $E \subset \mathbf{R}$, we define it in $\mathbf{R} \setminus E$ as zero. If the weight does not appear explicitly, we mean that we are using the weight 1. Given $0 < m < k$, a vectorial weight w and a closed set E, we denote by $W^{k,\infty}(E, w)$ the space $W^{k,\infty}(\Delta \cap E, w|_E)$ and by $W^{k-m,\infty}(\Delta, w)$ the space $W^{k-m,\infty}(\Delta, (w_m, ..., w_k))$. We denote by $\operatorname{supp} v$ the support of the measure $v(x)\,dx$, i.e., the intersection of every closed set $E \subseteq \mathbf{R}$ verifying $\int_{\mathbf{R}\setminus E} v = 0$. If A is a Borel set, $|A|$, χ_A, $int(A)$, and $\bar{A}$ denote, respectively, the Lebesgue measure, the characteristic function, the interior and the closure of A. If I, U are subsets of $\mathbf{R}$, the symbol $\partial_I U$ denotes the relative boundary of U in I. By $f^{(j)}$ we mean the jth distributional derivative of f. P denotes the set of polynomials. We say that an n-dimensional vector satisfies a one-dimensional property if each coordinate satisfies this property. Finally, the constants in the formulae can vary from line to line and even in the same line.

The outline of the paper is as follows. Section 2 is dedicated to the proof of the theorems in the case $k=0$. Section 3 presents most of the definitions we need to state the results in Sections 5 and 6. In Section 4 we collect the technical results of [RARP1], [RARP2], and [R1] that we need. We prove the theorems for $k \geqslant 1$ in Sections 5 and 6. The results for non-bounded intervals are proved in Section 6.

2. APPROXIMATION IN $L^{\infty}(I, W)$

DEFINITION 2.1. Given a measurable set A, we define the essential closure of A as the set

$$\operatorname{ess\,cl} A := \{x \in \mathbf{R} \colon |A \cap (x-\delta, x+\delta)| > 0,\ \forall \delta > 0\}.$$

DEFINITION 2.2. If A is a measurable set, f is a function defined in A with real values and $a \in \operatorname{ess\,cl} A$, we say that $\operatorname{ess\,lim}_{x \in A, x \to a} f(x) = l \in \mathbf{R}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ for almost every $x \in A \cap (a-\delta, a+\delta)$. In a similar way we can define $\operatorname{ess\,lim}_{x \in A, x \to a} f(x) = \infty$ and $\operatorname{ess\,lim}_{x \in A, x \to a} f(x) = -\infty$. We define the essential limit superior and the essential limit inferior in A as follows:

$$\operatorname*{ess\,lim\,sup}_{x \in A,\, x \to a} f(x) := \inf_{\delta > 0} \operatorname*{ess\,sup}_{x \in A \cap (a-\delta, a+\delta)} f(x),$$

$$\operatorname*{ess\,lim\,inf}_{x \in A,\, x \to a} f(x) := \sup_{\delta > 0} \operatorname*{ess\,inf}_{x \in A \cap (a-\delta, a+\delta)} f(x).$$

If we do not specify the set A we are assuming that $A = \mathbf{R}$.

Remarks.

1. The essential limit inferior (or superior) of a function f does not change if we modify f in a set of zero Lebesgue measure.

2. We have

$$\operatorname*{ess\,lim\,sup}_{x\in A,\, x\to a} f(x) \geqslant \operatorname*{ess\,lim\,inf}_{x\in A,\, x\to a} f(x),$$

$\operatorname*{ess\,lim}_{x\in A,\, x\to a} f(x) = l$ if and only if $\operatorname*{ess\,lim\,sup}_{x\in A,\, x\to a} f(x) = \operatorname*{ess\,lim\,inf}_{x\in A,\, x\to a} f(x) = l.$

3. We impose the condition $a \in \operatorname{ess\,cl} A$ in order to have the unicity of the essential limit. If we do not have this condition, then every real number is an essential limit for any function f.

DEFINITION 2.3. Given an interval I and a weight w in I we say that $a \in \bar{I}$ is a singularity of w (or singular for w) in I if

$$\operatorname*{ess\,lim\,inf}_{x\in I,\, x\to a} w(x) = 0.$$

We say that a singularity a of w is of type 1 if

$$\operatorname*{ess\,lim}_{x\in I,\, x\to a} w(x) = 0.$$

In other cases we say that a is a singularity of type 2.

Remark. The set of points which are not singular for w in I is a relative open set in I.

LEMMA 2.1. *Let us consider an interval I, a weight w in I, and a point $a \in \bar{I}$ which is not singular for w in I. Then there exists $\delta > 0$ such that every function in the closure of $C(\mathbf{R})$ with the norm $L^\infty(I, w)$ belongs to $C(\bar{I} \cap [a-\delta, a+\delta])$.*

Remark. Observe that every function in $C(\bar{I})$ can be extended to a function in $C(\mathbf{R})$; therefore, the closures of $C(\mathbf{R})$ and $C(\bar{I})$ with the norm $L^\infty(I, w)$ are the same. Recall that we identify functions which are equal almost everywhere.

Proof. We have that

$$\sup_{\delta>0} \operatorname*{ess\,inf}_{x\in I\cap(a-\delta,\, a+\delta)} w(x) = l > 0.$$

Therefore there exists $\delta > 0$ with

$$\operatorname*{ess\,inf}_{x\in I\cap(a-\delta,\, a+\delta)} w(x) > \frac{l}{2} > 0.$$

Hence, we have

$$\|g\|_{L^\infty(I\cap(a-\delta,\,a+\delta),\,w)} \geqslant \frac{l}{2}\,\|g\|_{L^\infty(I\cap(a-\delta,\,a+\delta))} = \frac{l}{2}\max_{x\in\bar{I}\cap[a-\delta,\,a+\delta]}|g(x)|,$$

for every $g\in C(\mathbf{R})$. This inequality gives the lemma, since if f is the limit of functions $\{g_n\}\subset C(\mathbf{R})$ with the norm in $L^\infty(I\cap(a-\delta,a+\delta),w)$, it can be modified in a set of zero Lebesgue measure in such a way that it is the uniform limit of $\{g_n\}$ in $\bar{I}\cap[a-\delta,a+\delta]$.

LEMMA 2.2. *Let us consider an interval I, a weight w in I and a singular point a of w in I of type* 1. *Then every function f in the closure of $C(\mathbf{R})$ with the norm $L^\infty(I,w)$ verifies*

$$\operatorname*{ess\,lim}_{x\in I,\,x\to a} f(x)\,w(x)=0. \tag{2.1}$$

Proof. Let us assume that (2.1) is not true, i.e.,

$$\operatorname*{ess\,lim\,sup}_{x\in I,\,x\to a}|f(x)|\,w(x)=l>0.$$

Therefore for every $\delta>0$ we have

$$\operatorname*{ess\,sup}_{x\in I\cap(a-\delta,\,a+\delta)}|f(x)|\,w(x)\geqslant l>0.$$

Since a is of type 1 we deduce

$$\operatorname*{ess\,lim}_{x\in I\cap(a-\delta,\,a+\delta)}|g(x)|\,w(x)=0,$$

for every $g\in C(\mathbf{R})$. This implies that for each $g\in C(\mathbf{R})$ and $\varepsilon>0$ there exists $\delta>0$ with

$$\operatorname*{ess\,sup}_{x\in I\cap(a-\delta,\,a+\delta)}|g(x)|\,w(x)\leqslant\varepsilon.$$

Consequently, for this $\delta>0$ we have

$$\begin{aligned}\|f-g\|_{L^\infty(I,w)} &\geqslant \|f-g\|_{L^\infty(I\cap(a-\delta,\,a+\delta),\,w)}\\ &\geqslant \|f\|_{L^\infty(I\cap(a-\delta,\,a+\delta),\,w)}-\|g\|_{L^\infty(I\cap(a-\delta,\,a+\delta),\,w)}\geqslant l-\varepsilon,\end{aligned}$$

for every $\varepsilon>0$ and $g\in C(\mathbf{R})$. Hence we have

$$\|f-g\|_{L^\infty(I,w)}\geqslant l>0,$$

for every $g \in C(\mathbf{R})$. This implies that f can not be approximated by functions in $C(\mathbf{R})$ with the norm $L^\infty(I, w)$.

LEMMA 2.3. *Let us consider an interval I and a weight $w \in L^\infty(I)$. Denote by S the set of singular points of w in I. Assume that $a \in S$ is of type 1 and $|S| = 0$. Then, for any fixed $\varepsilon > 0$ and $f \in C(I \setminus S) \cap L^\infty(I, w)$ with* $\operatorname{ess\,lim}_{x \in I, x \to a} f(x)\, w(x) = 0$, *there exist a relative open interval U in I with $a \in U$ and $\partial_I U \subset I \setminus S$ (and $U \subset int(I)$ if $a \in int(I)$) and a function $g \in L^\infty(I, w) \cap C(\bar{U})$ such that $g = f$ in $I \setminus U$, $\|f - g\|_{L^\infty(I, w)} < \varepsilon$ (and $\|f - g\|_{L^1(I)} < \varepsilon$ if $f \in L^1(I)$). Furthermore, we can choose g with the additional condition $g(a) = 0$ or even $g(a) = \lambda$ for any fixed $\lambda \in \mathbf{R}$.*

Proof. Without loss of generality we can assume that a is an interior point of I, since the case $a \in \partial I$ is simpler. Take n such that $[a - 1/n, a + 1/n] \subset int(I)$. Since $|S| = 0$, there exist $y_n \in (a, a + 1/n) \setminus S$ and $x_n \in (a - 1/n, a) \setminus S$ verifying

$$|f(y_n)| \leqslant 2^{-n} + \operatorname*{ess\,inf}_{x \in [a, a + 1/n]} |f(x)|, \qquad |f(x_n)| \leqslant 2^{-n} + \operatorname*{ess\,inf}_{x \in [a - 1/n, a]} |f(x)|.$$

Let us define now the function f_n (which is continuous in an open neighbourhood of $[x_n, y_n]$, since $x_n, y_n \notin S$) as

$$f_n(x) := \begin{cases} \dfrac{x - a}{x_n - a}\, f(x_n) & \text{if} \quad x \in [x_n, a], \\[2ex] \dfrac{x - a}{y_n - a}\, f(y_n) & \text{if} \quad x \in [a, y_n], \\[2ex] f(x) & \text{if} \quad x \in I \setminus [x_n, y_n]. \end{cases}$$

Observe that $|f_n(x)| \leqslant 2^{-n} + |f(x)|$ for almost every $x \in [x_n, y_n]$. Hence

$$\|f - f_n\|_{L^\infty(I, w)} = \|f - f_n\|_{L^\infty([x_n, y_n], w)} \leqslant 2\, \|f\|_{L^\infty([x_n, y_n], w)} + 2^{-n}\, \|w\|_{L^\infty(I)},$$

and this last expression goes to 0 as $n \to \infty$, since $\operatorname{ess\,lim}_{x \in I, x \to a} f(x)\, w(x) = 0$. If $f \in L^1(I)$, we also have

$$\|f - f_n\|_{L^1(I)} = \|f - f_n\|_{L^1([x_n, y_n])} \leqslant 2\, \|f\|_{L^1([x_n, y_n])} + 2^{-n}(y_n - x_n),$$

and this expression goes to 0 as $n \to \infty$. Observe that $f_n(a) = 0$; it is easy to modify f_n in a small neighbourhood of a in order to have $f_n(a) = \lambda$, for fixed $\lambda \in \mathbf{R}$. This finishes the proof of the lemma.

LEMMA 2.4. *If A is a measurable set, we have:*

(1) $\operatorname{ess\,cl} A$ *is a closed set contained in* $\overline{A}$.

(2) $|A \setminus \operatorname{ess\,cl} A| = 0$.

(3) *If f is a measurable function in $A \cup \operatorname{ess\,cl} A$, $a \in \operatorname{ess\,cl} A$ and there exists* $\operatorname{ess\,lim}_{x \in \operatorname{ess\,cl} A,\, x \to a} f(x)$, *then there exists* $\operatorname{ess\,lim}_{x \in A,\, x \to a} f(x)$ *and*

$$\operatorname*{ess\,lim}_{x \in A,\, x \to a} f(x) = \operatorname*{ess\,lim}_{x \in \operatorname{ess\,cl} A,\, x \to a} f(x).$$

(4) *If $|A| > 0$ and f is a continuous function in* $\mathbf{R}$ *we have*

$$\|f\|_{L^\infty(A)} = \sup_{x \in \operatorname{ess\,cl} A} |f(x)|\,.$$

Proof. (1) is direct.

(2) is a consequence of the Lebesgue differentiation theorem, since we have

$$\lim_{\delta \to 0} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} \chi_A = 1,$$

for almost every $x \in A$, and this implies $|A \cap (x-\delta, x+\delta)| > 0$ for almost every $x \in A$ and every $\delta > 0$.

Assume now that $\operatorname{ess\,lim}_{x \in \operatorname{ess\,cl} A,\, x \to a} f(x) = l \in \mathbf{R}$. Consequently, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for almost every $x \in \operatorname{ess\,cl} A \cap (a-\delta, a+\delta)$ we have $|f(x) - l| < \varepsilon$. Since $|A \setminus \operatorname{ess\,cl} A| = 0$, we have $|f(x) - l| < \varepsilon$, for almost every $x \in A \cap (a-\delta, a+\delta)$. This gives (3) if $l \in \mathbf{R}$. The case $l = \pm\infty$ is similar.

The statement (2) gives

$$\|f\|_{L^\infty(A)} \leqslant \|f\|_{L^\infty(\operatorname{ess\,cl} A)} \leqslant \sup_{x \in \operatorname{ess\,cl} A} |f(x)|\,.$$

We have $|f(x)| \leqslant \|f\|_{L^\infty(A)}$ for almost every $x \in A$. Then $|f(x)| \leqslant \|f\|_{L^\infty(A)}$ for every $x \in \operatorname{ess\,cl} A$, since f is continuous. Therefore

$$\sup_{x \in \operatorname{ess\,cl} A} |f(x)| \leqslant \|f\|_{L^\infty(A)}\,.$$

These two inequalities give (4).

LEMMA 2.5. *Let us consider an interval I, a weight w in I, and $a \in \bar{I}$. If* $\operatorname{ess\,lim\,sup}_{x \in I,\, x \to a} w(x) = l > 0$, *then for every function f in the closure of $C(\mathbf{R}) \cap L^\infty(I, w)$ with the norm $L^\infty(I, w)$ there exists the finite limit*

$$\operatorname*{ess\,lim}_{w(x) \geqslant \varepsilon,\, x \to a} f(x), \qquad \textit{for every} \quad 0 < \varepsilon < l.$$

Proof. We have for every $\delta > 0$

$$\operatorname*{ess\,sup}_{x \in I \cap (a-\delta,\, a+\delta)} w(x) \geqslant l > 0,$$

and then

$$|\{x \in I \cap (a-\delta, a+\delta) : w(x) \geqslant \varepsilon\}| > 0,$$

for every $\delta > 0$ and $0 < \varepsilon < l$. This implies that a belongs to ess cl A_ε, where $A_\varepsilon := \{x \in I : w(x) \geqslant \varepsilon\}$.

If $g \in C(\mathbf{R}) \cap L^\infty(I, w)$, $0 < \varepsilon < l$, and $\delta > 0$, we have

$$\varepsilon \, \|g\|_{L^\infty(A_\varepsilon \cap [a-\delta,\, a+\delta])} \leqslant \|g\|_{L^\infty(A_\varepsilon \cap [a-\delta,\, a+\delta],\, w)}.$$

Since ess cl $(A_\varepsilon \cap [a-\delta, a+\delta])$ is a compact set and $g \in C(\mathbf{R}) \cap L^\infty(I, w)$, Lemma 2.4 (4) gives

$$\varepsilon \max_{x \in \text{ess cl } (A_\varepsilon \cap [a-\delta,\, a+\delta])} |g(x)| \leqslant \|g\|_{L^\infty(A_\varepsilon \cap [a-\delta,\, a+\delta],\, w)}.$$

Consequently, if $\{g_n\} \subset C(\mathbf{R}) \cap L^\infty(I, w)$ converges to f in $L^\infty(I, w)$, then $\{g_n\}$ converges to f uniformly in ess cl $(A_\varepsilon \cap [a-\delta, a+\delta])$ and $f \in C($ess cl $(A_\varepsilon \cap [a-\delta, a+\delta]))$ for every $\delta > 0$. Therefore $f \in C($ess cl $A_\varepsilon)$. This fact and Lemma 2.4 (3) give that, for $0 < \varepsilon < l$, there exists

$$\operatorname*{ess\,lim}_{x \in A_\varepsilon,\, x \to a} f(x) = \operatorname*{ess\,lim}_{x \in \text{ess cl } A_\varepsilon,\, x \to a} f(x) = \lim_{x \in \text{ess cl } A_\varepsilon,\, x \to a} f(x).$$

LEMMA 2.6. *Let us consider an interval I, a weight w in I, and a singular point a of w in I. Then every function f in the closure of $C(\mathbf{R}) \cap L^\infty(I, w)$ with the norm $L^\infty(I, w)$ verifies*

$$\inf_{\varepsilon > 0} \Big(\operatorname*{ess\,lim\,sup}_{w(x) < \varepsilon,\, x \to a} |f(x)| \, w(x) \Big) = 0. \tag{2.2}$$

Proof. Observe first that $a \in$ ess cl$(\{x \in I : w(x) < \varepsilon\})$ for every $\varepsilon > 0$, since a is singular for w in I. Let us assume that (2.2) is not true, i.e.,

$$\operatorname*{ess\,lim\,sup}_{x \in A_\varepsilon^c,\, x \to a} |f(x)| \, w(x) \geqslant l > 0,$$

for every $\varepsilon > 0$, where $A_\varepsilon := \{x \in I : w(x) \geqslant \varepsilon\}$ and $A_\varepsilon^c := I \backslash A_\varepsilon$. Therefore for every $\varepsilon, \delta > 0$ we have

$$\operatorname*{ess\,sup}_{x \in A_\varepsilon^c \cap (a-\delta, a+\delta)} |f(x)| \, w(x) \geqslant l > 0.$$

For each $g \in C(\mathbf{R}) \cap L^\infty(I, w)$, $\varepsilon > 0$, and $\delta > 0$, we have

$$\|g\|_{L^\infty(A_\varepsilon^c \cap (a-\delta, a+\delta), w)} \leqslant \varepsilon \, \|g\|_{L^\infty(I \cap (a-\delta, a+\delta))} < \infty .$$

Consequently

$$\begin{aligned} \|f-g\|_{L^\infty(I, w)} &\geqslant \|f-g\|_{L^\infty(A_\varepsilon^c \cap (a-\delta, a+\delta), w)} \\ &\geqslant \|f\|_{L^\infty(A_\varepsilon^c \cap (a-\delta, a+\delta), w)} - \|g\|_{L^\infty(A_\varepsilon^c \cap (a-\delta, a+\delta), w)}, \end{aligned}$$

and therefore

$$\|f-g\|_{L^\infty(I, w)} \geqslant l - \varepsilon \, \|g\|_{L^\infty(I \cap (a-\delta, a+\delta))} ,$$

for every $g \in C(\mathbf{R}) \cap L^\infty(I, w)$ and $\delta, \varepsilon > 0$. Hence we obtain

$$\|f-g\|_{L^\infty(I, w)} \geqslant l > 0 ,$$

for every $g \in C(\mathbf{R}) \cap L^\infty(I, w)$. This implies that f cannot be approximated by functions in $C(\mathbf{R}) \cap L^\infty(I, w)$.

LEMMA 2.7. *Let us consider an interval I, a weight w in I, and $a \in \bar{I}$. If* $\operatorname{ess\,lim}_{x \in I, x \to a} w(x) = 0$ *and* $\inf_{\varepsilon > 0} (\operatorname{ess\,lim\,sup}_{w(x) < \varepsilon, x \to a} |f(x)| \, w(x)) = 0$, *then we have* $\operatorname{ess\,lim}_{x \in I, x \to a} f(x) \, w(x) = 0$.

Proof. For each $\eta > 0$ there exist $\varepsilon, \delta_1 > 0$ such that

$$\operatorname*{ess\,sup}_{w(x) < \varepsilon, \, x \in I \cap (a-\delta_1, a+\delta_1)} |f(x)| \, w(x) < \eta.$$

We also have that there exists $\delta_2 > 0$ such that $w(x) < \varepsilon$ for almost every $x \in I \cap (a-\delta_2, a+\delta_2)$. If we take $\delta := \min(\delta_1, \delta_2)$, we obtain

$$\operatorname*{ess\,sup}_{x \in I \cap (a-\delta, a+\delta)} |f(x)| \, w(x) \leqslant \operatorname*{ess\,sup}_{w(x) < \varepsilon, \, x \in I \cap (a-\delta_1, a+\delta_1)} |f(x)| \, w(x) < \eta ,$$

and this finishes the proof.

LEMMA 2.8. *Let us consider an interval I and a weight $w \in L^\infty(I)$. Denote by S the set of singular points of w in I. Assume that $a \in S$ and $|S| = 0$. Then, for any fixed $\eta > 0$ and $f \in C(I \backslash S) \cap L^\infty(I, w)$ such that*

(a) $\inf_{\varepsilon>0}(\operatorname{ess\,lim\,sup}_{w(x)<\varepsilon,\, x\to a} |f(x)|\, w(x))=0$,

(b) *there exists the finite limit* $\operatorname{ess\,lim}_{w(x)\geqslant\varepsilon,\, x\to a} f(x)$, *for* $\varepsilon>0$ *small enough,*

there exist a relative open interval U *in* I *with* $a\in U$ *and* $\partial_I U\subset I\backslash S$ (*and* $U\subset int(I)$ *if* $a\in int(I)$) *and a function* $g\in L^\infty(I,w)\cap C(\bar{U})$ *with* $g=f$ *in* $I\backslash U$, $\|f-g\|_{L^\infty(I,w)}<\eta$ (*and* $\|f-g\|_{L^1(I)}<\eta$ *if* $f\in L^1(I)$). *Furthermore, we can choose* g *with the additional condition* $g(a)=\operatorname{ess\,lim}_{w(x)\geqslant\varepsilon,\, x\to a} f(x)$, *for* $\varepsilon>0$ *small enough.*

Proof. If a is of type 1, Lemmas 2.3 and 2.7 give the result. Assume now that a is of type 2. Without loss of generality we can assume that a is an interior point of I, since the case $a\in\partial I$ is simpler.

We consider first the case $\operatorname{ess\,lim\,sup}_{x\to a^+} w(x)>0$ and $\operatorname{ess\,lim\,sup}_{x\to a^-} w(x)>0$. For each natural number n, let us choose $\varepsilon_n>0$ with $\lim_{n\to\infty}\varepsilon_n=0$ and

$$\operatorname*{ess\,lim\,sup}_{w(x)<\varepsilon_n,\, x\to a} |f(x)|\, w(x)<\frac{1}{n}.$$

Let us consider now $\delta_n>0$ with $\lim_{n\to\infty}\delta_n=0$ and

$$\operatorname*{ess\,sup}_{x\in(a-\delta_n,\, a+\delta_n)\cap A^c_{\varepsilon_n}} |f(x)|\, w(x)<\frac{1}{n}, \tag{2.3}$$

where $A_\varepsilon := \{x\in I : w(x)\geqslant\varepsilon\}$ and $A_\varepsilon{}^c := I\backslash A_\varepsilon$. We define $l := \operatorname{ess\,lim}_{x\in A_\varepsilon,\, x\to a} f(x)$, for any $\varepsilon>0$ small enough. We can take δ_n with the additional property $|f(x)-l|<1/n$ for almost every $x\in(a-\delta_n, a+\delta_n)\cap A_{\varepsilon_n}$. Let us choose $\gamma_n\in(a, a+\delta_n)\backslash S$ and $\gamma'_n\in(a-\delta_n, a)\backslash S$ with $|f(\gamma_n)-l|<1/n$ and $|f(\gamma'_n)-l|<1/n$. We define the functions $a_n(x)$ and $b_n(x)$ in $[\gamma'_n, \gamma_n]$ as follows:

$$a_n(x) := \begin{cases} l+(x-a)\dfrac{\min\{l, f(\gamma_n)\}-l}{\gamma_n-a} & \text{if } x\in[a,\gamma_n], \\[2ex] l+(x-a)\dfrac{\min\{l, f(\gamma'_n)\}-l}{\gamma'_n-a} & \text{if } x\in[\gamma'_n, a], \end{cases}$$

and

$$b_n(x) := \begin{cases} l+(x-a)\dfrac{\max\{l, f(\gamma_n)\}-l}{\gamma_n-a} & \text{if } x\in[a,\gamma_n], \\[2ex] l+(x-a)\dfrac{\max\{l, f(\gamma'_n)\}-l}{\gamma'_n-a} & \text{if } x\in[\gamma'_n, a]. \end{cases}$$

Now we can define the functions $g_n \in L^\infty(I, w) \cap C([\gamma'_n, \gamma_n])$ in the following way:

$$g_n(x) := \begin{cases} a_n(x) & \text{if} \quad x \in [\gamma'_n, \gamma_n] \text{ and } f(x) \leqslant a_n(x)\,, \\ b_n(x) & \text{if} \quad x \in [\gamma'_n, \gamma_n] \text{ and } f(x) \geqslant b_n(x)\,, \\ f(x) & \text{in other case}\,. \end{cases}$$

Observe that $a_n(x) \leqslant g_n(x) \leqslant b_n(x)$, $|a_n(x) - l| < 1/n$, and $|b_n(x) - l| < 1/n$, for every $x \in [\gamma'_n, \gamma_n]$. Therefore $|g_n(x) - l| < 1/n$ for $x \in [\gamma'_n, \gamma_n]$ and

$$\|f - g_n\|_{L^\infty([\gamma'_n, \gamma_n] \cap A_{\varepsilon_n}, w)} \leqslant \frac{2}{n} \|w\|_{L^\infty(I)}\,. \tag{2.4}$$

We prove now $\lim_{n \to \infty} \|f - g_n\|_{L^\infty([\gamma'_n, \gamma_n], w)} = 0$. The facts

$$\|g_n\|_{L^\infty([\gamma'_n, \gamma_n] \cap A^c_{\varepsilon_n}, w)} \leqslant \left(|l| + \frac{1}{n}\right) \varepsilon_n\,,$$

and (2.3) give

$$\|f - g_n\|_{L^\infty([\gamma'_n, \gamma_n] \cap A^c_{\varepsilon_n}, w)} < \frac{1}{n} + \left(|l| + \frac{1}{n}\right) \varepsilon_n\,.$$

This inequality and (2.4) give

$$\|f - g_n\|_{L^\infty([\gamma'_n, \gamma_n], w)} < \frac{1}{n} + \left(|l| + \frac{1}{n}\right) \varepsilon_n + \frac{2}{n} \|w\|_{L^\infty(I)}\,.$$

If $f \in L^1(I)$, we also have

$$\begin{aligned} \|f - g_n\|_{L^1(I)} &= \|f - g_n\|_{L^1([\gamma'_n, \gamma_n])} \\ &\leqslant \|f\|_{L^1([\gamma'_n, \gamma_n])} + \left(|l| + \frac{1}{n}\right)(\gamma_n - \gamma'_n). \end{aligned}$$

This finishes the proof in this case.

If $\operatorname{ess\,lim\,sup}_{x \to a^+} w(x) > 0$ and $\operatorname{ess\,lim\,sup}_{x \to a^-} w(x) = 0$, we only need to consider the functions g_n for $x > a$ and the functions f_n in the proof of Lemma 2.3 for $x < a$ (recall that we can choose f_n with $f_n(a) = l$).

The case $\operatorname{ess\,lim\,sup}_{x \to a^+} w(x) = 0$ and $\operatorname{ess\,lim\,sup}_{x \to a^-} w(x) > 0$ is symmetric.

The following result is direct.

PROPOSITION 2.1. *Let us consider a sequence of closed intervals $\{I_n\}_{n \in \Lambda}$ such that for each $n \in \Lambda$ there exists an open neighbourhood of I_n which does* *not intersect $\bigcup_{m\neq n} I_m$. Denote by J the union $J := \bigcup_n I_n$. Let us consider a weight w in J. Then we have*

$$\overline{C(J)\cap L^\infty(J,w)} = \bigcap_n \overline{C(I_n)\cap L^\infty(I_n,w)},$$

where the closures are taken in L^∞ with respect to w, in the corresponding interval.

We also have a similar result for contiguous intervals.

PROPOSITION 2.2. *Let us consider an interval I and a weight $w \in L^\infty_{loc}(I)$. Let us consider an increasing sequence of real numbers $\{a_n\}_{n\in\Lambda}$, where Λ is either $\mathbf{Z}^+$, $\mathbf{Z}^-$, $\mathbf{Z}$, or $\{1,2,...,N\}$ for some $N\in\mathbf{N}$ such that $I=\bigcup_n[a_n,a_{n+1}]$ and a_n is not singular for w in I if a_n is in the interior of I. Then we have*

$$\begin{aligned}
&\overline{C^\infty(I)\cap L^\infty(I,w)}\\
&\quad= \overline{C(I)\cap L^\infty(I,w)}\\
&\quad= \left\{ f\in \bigcap_{n\in\Lambda} \overline{C([a_n,a_{n+1}])} : f \text{ is continuous in each } a_n\in \operatorname{int}(I)\right\}\\
&\quad= \left\{ f\in \bigcap_{n\in\Lambda} \overline{C^\infty([a_n,a_{n+1}])} : f \text{ is continuous in each } a_n\in \operatorname{int}(I)\right\},
\end{aligned}$$

where the closures are taken in L^∞ with respect to w, in the corresponding interval.

Remark. We can ensure $\overline{C^\infty(I)\cap L^\infty(I,w)} = \overline{C^\infty(\mathbf{R})\cap L^\infty(I,w)}$ if I is closed. The same is obviously true for $C(I)$ instead of $C^\infty(I)$.

Proof. The third equality is true since $\overline{C^\infty([a_n,a_{n+1}])} = \overline{C([a_n,a_{n+1}])}$ is a direct consequence of Weierstrass' theorem and $w\in L^\infty([a_n,a_{n+1}])$.

We are going to see that the closure of $C^\infty(I)\cap L^\infty(I,w)$ and $C(I)\cap L^\infty(I,w)$ with the norm $L^\infty(I,w)$ is the same. It is enough to prove that every $f\in C(I)$ can be approximated by functions in $C^\infty(I)$ with the norm $L^\infty(I,w)$. We can assume that $\Lambda=\mathbf{Z}$, since the argument in the other cases is simpler. Given $\varepsilon>0$ and $f\in C(I)$, for each $n\in\mathbf{Z}$, there exists a function $g_n\in C^\infty(\mathbf{R})$ with $\|f-g_n\|_{L^\infty([a_{2n-1},a_{2n+2}],w)}<\varepsilon/2$. Let us consider functions $\theta_n\in C^\infty(\mathbf{R})$ with $\theta_n=0$ in $(-\infty,a_{2n-1}]$, $\theta_n=1$ in $[a_{2n},\infty)$ and $0\leqslant\theta_n\leqslant 1$.

We define now a function $g \in C^\infty(I)$ by $g(x) := (1-\theta_n(x))\, g_{n-1}(x) + \theta_n(x)\, g_n(x)$, if $x \in [a_{2n-1}, a_{2n+1}]$. We have

$$\|f-g\|_{L^\infty([a_{2n-1}, a_{2n}], w)} \leqslant \|(1-\theta_n)(f-g_{n-1})\|_{L^\infty([a_{2n-1}, a_{2n}], w)} + \|\theta_n(f-g_n)\|_{L^\infty([a_{2n-1}, a_{2n}], w)} < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

$$\|f-g\|_{L^\infty([a_{2n}, a_{2n+1}], w)} = \|f-g_n\|_{L^\infty([a_{2n}, a_{2n+1}], w)} < \varepsilon/2\,,$$

and this implies $\|f-g\|_{L^\infty(I, w)} \leqslant \varepsilon$.

In order to see the second equality, observe that the ideas above give that the result is true if it is true for the set $\Lambda = \{1, 2, 3\}$. Let us consider $f \in \overline{C([a_1, a_2])} \cap \overline{C([a_2, a_3])}$ and continuous in a_2. Given $m \in \mathbf{N}$ there exist functions $g_m^1 \in C([a_1, a_2])$ and $g_m^2 \in C([a_2, a_3])$ with $\|f-g_m^1\|_{L^\infty([a_1, a_2], w)} + \|f-g_m^2\|_{L^\infty([a_2, a_3], w)} < 1/m$.

In order to finish the proof it is enough to construct a function $g_m \in C([a_1, a_3])$ satisfying the inequality $\|f-g_m\|_{L^\infty([a_1, a_3], w)} < c/m$, where c is a constant independent of m. We know that there exist positive constants δ, c_1, and c_2 such that $[a_2-\delta, a_2+\delta] \subseteq [a_1, a_3]$, $|f(x)-f(a_2)| < 1/m$ if $|x-a_2| \leqslant \delta$ and $0 < c_1^{-1} \leqslant w(x) \leqslant c_2$ for almost every $x \in [a_2-\delta, a_2+\delta]$. Lemma 2.1 gives that $f \in C([a_2-\delta, a_2+\delta])$ and then

$$|f(x)-g_m^1(x)| < c_1/m, \qquad \text{for every} \quad x \in [a_2-\delta, a_2],$$

$$|f(x)-g_m^2(x)| < c_1/m, \qquad \text{for every} \quad x \in [a_2, a_2+\delta],$$

and consequently

$$|g_m^1(x)-f(a_2)| < (c_1+1)/m, \qquad \text{for every} \quad x \in [a_2-\delta, a_2],$$

$$|g_m^2(x)-f(a_2)| < (c_1+1)/m, \qquad \text{for every} \quad x \in [a_2, a_2+\delta].$$

Let us define g_m^0 as the function whose graph is the segment joining the points $(a_2-\delta, g_m^1(a_2-\delta))$ and $(a_2+\delta, g_m^2(a_2+\delta))$. Then we have

$$|g_m^0(x)-f(a_2)| < (c_1+1)/m, \qquad \text{for every} \quad x \in [a_2-\delta, a_2+\delta],$$

$$|g_m^0(x)-f(x)| < (c_1+2)/m, \qquad \text{for every} \quad x \in [a_2-\delta, a_2+\delta],$$

$$\|g_m^0-f\|_{L^\infty([a_2-\delta, a_2+\delta], w)} < c_2(c_1+2)/m.$$

If we define the function $g_m \in C([a_1, a_3])$ by

$$g_m(x) := \begin{cases} g_m^1(x), & \text{if} \quad x \in [a_1, a_2-\delta], \\ g_m^0(x), & \text{if} \quad x \in [a_2-\delta, a_2+\delta], \\ g_m^2(x)\,, & \text{if} \quad x \in [a_2+\delta, a_3], \end{cases}$$

we have

$$\|f-g_m\|_{L^\infty([a_1,a_3],w)}<(c_2(c_1+2)+1)/m.$$

This finishes the proof of Proposition 2.2.

PROPOSITION 2.3. *Let us consider a closed interval I and a weight $w\in L^\infty_{loc}(I)$ such that the set S of singular points of w in I has zero Lebesgue measure. Then we have $\overline{C^\infty(\mathbf{R})\cap L^\infty(I,w)}=\overline{C(\mathbf{R})\cap L^\infty(I,w)}=H$, with*

$$H:=\{f\in C(I\backslash S)\cap L^\infty(I,w):$$

$$\text{for each } a\in S,\ \inf_{\varepsilon>0}\ (\operatorname{ess}\limsup_{w(x)<\varepsilon,\,x\to a}|f(x)|\,w(x))=0$$

and, if a is of type 2,

$$\text{there exists the finite limit } \operatorname*{ess\,lim}_{w(x)\geqslant\varepsilon,\,x\to a} f(x),$$

$$\text{for } \varepsilon>0 \text{ small enough}\},$$

where the closures are taken in $L^\infty(I,w)$. Furthermore, if I is compact we also have $P^{0,\infty}(I,w)=H$.

If $f\in H\cap L^1(I)$, I is compact, and S is countable, we can approximate f by polynomials with the norm $\|\cdot\|_{L^\infty(I,w)}+\|\cdot\|_{L^1(I)}$.

Proof. Lemmas 2.1, 2.2, 2.7, 2.5, and 2.6 give that H contains $\overline{C(\mathbf{R})\cap L^\infty(I,w)}$. In order to see that H is contained in $\overline{C(\mathbf{R})\cap L^\infty(I,w)}$, assume first that I is compact; then $w\in L^\infty(I)$. Fix $\varepsilon>0$ and $f\in H$. Lemma 2.8 gives that for each $a\in S$ there exist a relative open interval U_a in I with $a\in U_a$ and $\partial_I U_a\subset I\backslash S$ (and $U_a\subset int(I)$ if $a\in int(I)$) and a function $g_a\in L^\infty(I,w)\cap C(\bar U_a)$ such that $g_a=f$ in $I\backslash U_a$ and $\|f-g_a\|_{L^\infty(I,w)}<\varepsilon$. The set S is compact since it is a closed set contained in the compact interval I. Therefore there exist $a_1,...,a_m\in S$ such that $S\subset U_{a_1}\cup\cdots\cup U_{a_m}$. Without loss of generality we can assume that $U_{a_1},...,U_{a_m}$ is minimal in the following sense: for each $i=1,...,m$ the set $\bigcup_{j\neq i}U_{a_j}$ does not contain to U_{a_i}.

Define $[\alpha_i,\beta_i]:=\bar U_{a_i}$. Assume that we have $U_{a_i}\cap U_{a_j}\neq\emptyset$, with $\alpha_i<\alpha_j$. The minimal property gives $\bar U_{a_i}\cap\bar U_{a_j}=[\alpha_j,\beta_i]$ and $[\alpha_j,\beta_i]\cap U_{a_k}=\emptyset$ for every $k\neq i,j$. We define the functions

$$g_{a_j,a_i}(x):=g_{a_i,a_j}(x):=\frac{\beta_i-x}{\beta_i-\alpha_j}\,g_{a_i}(x)+\frac{x-\alpha_j}{\beta_i-\alpha_j}\,g_{a_j}(x)\ .$$

Observe that $g_{a_i, a_j} \in C([\alpha_j, \beta_i])$ and satisfies $g_{a_i, a_j}(\alpha_j) = g_{a_i}(\alpha_j)$, $g_{a_i, a_j}(\beta_i) = g_{a_j}(\beta_i)$, and

$$\|g_{a_j, a_i} - f\|_{L^\infty([\alpha_j, \beta_i], w)} \leqslant \left\| \frac{\beta_i - x}{\beta_i - \alpha_j} (g_{a_i}(x) - f(x)) \right\|_{L^\infty([\alpha_j, \beta_i], w)} + \left\| \frac{x - \alpha_j}{\beta_i - \alpha_j} (g_{a_j}(x) - f(x)) \right\|_{L^\infty([\alpha_j, \beta_i], w)} < 2\varepsilon.$$

If we define the function $g \in L^\infty(I, w) \cap C(I)$ as

$$g(x) := \begin{cases} f(x) & \text{if } \ x \in I \setminus \bigcup_i U_{a_i} \\ g_{a_i}(x) & \text{if } \ x \in U_{a_i}, \ \ x \notin \bigcup_{j \neq i} U_{a_j} \\ g_{a_i, a_j}(x) & \text{if } \ x \in U_{a_i} \cap U_{a_j}, \end{cases}$$

we have $\|f - g\|_{L^\infty(I, w)} < 2\varepsilon$. If $f \in L^1(I)$ and S is countable, consider $\{a_1, a_2, ...\} = S$. If we take g_{a_n} with $\|f - g_{a_n}\|_{L^1(I)} < 2^{-n}\varepsilon$, it is direct that $\|f - g\|_{L^1(I)} < 2\varepsilon$. This finishes the proof in this case.

If I is not compact, we can choose an increasing sequence $\{a_n\}_{n \in \Lambda}$ of real numbers, where Λ is either $\mathbf{Z}^+$, $\mathbf{Z}^-$, or $\mathbf{Z}$ such that $I = \bigcup_n [a_n, a_{n+1}]$ and a_n is not singular for w in I if a_n is in the interior of I. We can take $\{a_n\}_{n \in \Lambda}$ with the following additional property: $\max_{n \in \Lambda} a_n = \max I$ if there exists $\max I$ and $\min_{n \in \Lambda} a_n = \min I$ if there exists $\min I$. The first part of the proof and Proposition 2.2 give the result.

We can reformulate this result as follows.

THEOREM 2.1. *Let us consider a closed interval I and a weight $w \in L^\infty_{loc}(I)$ such that the set S of singular points of w in I has zero Lebesgue measure. Then we have $\overline{C^\infty(\mathbf{R}) \cap L^\infty(I, w)} = \overline{C(\mathbf{R}) \cap L^\infty(I, w)} = H$, with*

$$H = \{ f \in C(I \setminus S) \cap L^\infty(I, w) : \text{for each } a \in S, \ \exists l_a \in \mathbf{R} \text{ such that } \operatorname*{ess\,lim}_{x \in I, x \to a} |f(x) - l_a| \, w(x) = 0 \},$$

where the closures are taken in $L^\infty(I, w)$. If $a \in S$ is of type 1, we can take as l_a any real number. If $a \in S$ is of type 2, $l_a = \operatorname{ess\,lim}_{w(x) \geqslant \varepsilon, x \to a} f(x)$ for $\varepsilon > 0$ small enough. Furthermore, if I is compact we also have $P^{0, \infty}(I, w) = H$.

If $f \in H \cap L^1(I)$, I is compact and S is countable, we can approximate f by polynomials with the norm $\|\cdot\|_{L^\infty(I, w)} + \|\cdot\|_{L^1(I)}$.

Remark. If S is the union of a finite number of intervals and a set of zero Lebesgue measure, Theorem 2.1 and Proposition 2.1 give $\overline{C^\infty(\mathbf{R}) \cap L^\infty(I, w)}$.

Proof. We only need to show the equivalence of the following conditions (a) and (b):

(a) for each $a \in S$,

(a.1) $\inf_{\varepsilon>0}(\text{ess}\limsup_{w(x)<\varepsilon,\, x \to a} |f(x)|\, w(x)\ ) = 0$,

(a.2) if a is of type 2, there exists the finite limit $l_a := \text{ess}\lim_{w(x) \geqslant \varepsilon,\, x \to a} f(x)$, for $\varepsilon > 0$ small enough,

(b) for each $a \in S$, there exists $l_a \in \mathbf{R}$ such that $\text{ess}\lim_{x \in I,\, x \to a} |f(x) - l_a|\, w(x) = 0$.

It is clear that (b) implies (a). Hypothesis (a.1) gives that for each $\eta > 0$, there exist $\varepsilon, \delta > 0$ with $\|f\|_{L^\infty([a-\delta, a+\delta] \cap \{w(x)<\varepsilon\}, w)} < \eta/3$ and $|l_a|\varepsilon < \eta/3$. By hypothesis (a.2) we can choose δ with the additional condition $\|f - l_a\|_{L^\infty([a-\delta, a+\delta] \cap \{w(x) \geqslant \varepsilon\}, w)} < \eta/3$. These inequalities imply

$$\|f - l_a\|_{L^\infty([a-\delta, a+\delta], w)} \leqslant \|f\|_{L^\infty([a-\delta, a+\delta] \cap \{w(x)<\varepsilon\}, w)} + |l_a|\, \varepsilon + \|f - l_a\|_{L^\infty([a-\delta, a+\delta] \cap \{w(x) \geqslant \varepsilon\}, w)} < \eta.$$

COROLLARY 2.1. *Let us consider a closed interval I and a weight $w \in L^\infty_{loc}(I)$ such that the set S of singular points of w in I has zero Lebesgue measure. If $f, g \in \overline{C(\mathbf{R}) \cap L^\infty(I, w)}$ and $\varphi \in C(I) \cap L^\infty(I)$, then we also have $|f|, f_+, f_-, \max(f, g), \min(f, g), \varphi f \in \overline{C(\mathbf{R}) \cap L^\infty(I, w)}$.*

Proof. The characterization of $\overline{C(\mathbf{R}) \cap L^\infty(I, w)}$ given in Theorem 2.1 or Proposition 2.3 implies the result for $|f|$ and φf. This fact and

$$\max(f, g) = \frac{f + g + |f - g|}{2}, \qquad \min(f, g) = \frac{f + g - |f - g|}{2}$$

gives the result for $\max(f, g)$ and $\min(f, g)$. The facts $f_+ = \max(f, 0)$, $f_- = \max(-f, 0)$ finish the proof.

3. PREVIOUS DEFINITIONS FOR SOBOLEV SPACES

Most of our results for $k \geqslant 1$ use tools of Sobolev spaces. We include here the definitions that we need in order to understand these tools.

First of all, we explain the definition of generalized Sobolev space in [RARP1] for the particular case $p = \infty$ (the definition in [RARP1] covers the cases $1 \leqslant p \leqslant \infty$, even if the weights are substituted by measures). One can think that the natural definition of weighted Sobolev space (the functions f with k weak derivatives satisfying $\|f^{(j)}\|_{L^\infty(w_j)} < \infty$ for $0 \leqslant j \leqslant k$) is a good one; however this is not true (see [KO] or [RARP1]). We start with some previous definitions.

Definition 3.1. We say that two functions u, v are comparable on the set A if there are positive constants c_1, c_2 such that $c_1 \leqslant u(x)/v(x) \leqslant c_2$ for almost every $x \in A$. We say that two norms $\|\cdot\|_1$, $\|\cdot\|_2$ in the vectorial space X are comparable if there are positive constants c_1, c_2 such that $c_1 \leqslant \|x\|_1/\|x\|_2 \leqslant c_2$ for every $x \in X$. We say that two vectorial weights are comparable if they are comparable on each component. (We use here the convention that $0/0 = 1$.)

In what follows the symbol $a \asymp b$ means that a and b are comparable for a and b functions or norms.

Obviously, the spaces $L^\infty(A, w)$ and $L^\infty(A, v)$ are the same and have comparable norms if w and v are comparable on A. Therefore, in order to study Sobolev spaces we can change a weight w by any comparable weight v.

Next, we shall define a class of weights which plays an important role in our results.

Definition 3.2. We say that a weight w belongs to $B_\infty([a, b])$ if $w^{-1} \in L^1([a, b])$. Also, if J is any interval we say that $w \in B_\infty(J)$ if $w \in B_\infty(I)$ for every compact interval $I \subseteq J$. We say that a weight belongs to $B_\infty(J)$, where J is a union of disjoint intervals $\bigcup_{i \in A} J_i$, if it belongs to $B_\infty(J_i)$, for $i \in A$.

Observe that if $v \geqslant w$ in J and $w \in B_\infty(J)$, then $v \in B_\infty(J)$.

Definition 3.3. We denote by $AC([a, b])$ the set of functions absolutely continuous in $[a, b]$, i.e. the functions $f \in C([a, b])$ such that $f(x) - f(a) = \int_a^x f'(t)\, dt$ for all $x \in [a, b]$. If J is any interval, $AC_{loc}(J)$ denotes the set of functions absolutely continuous in every compact subinterval of J.

Definition 3.4. Let us consider a vectorial weight $w = (w_0, ..., w_k)$. For $0 \leqslant j \leqslant k$ we define the open set

$$\Omega_j := \{x \in \mathbf{R} : \exists \text{ an open neighbourhood } V \text{ of } x \text{ with } w_j \in B_\infty(V)\}.$$

Observe that we always have $w_j \in B_\infty(\Omega_j)$ for any $0 \leqslant j \leqslant k$. In fact, Ω_j is the greatest open set U with $w_j \in B_\infty(U)$. Obviously, Ω_j depends on w, although w does not appear explicitly in the symbol Ω_j. It is easy to check that if $f^{(j)} \in L^\infty(\Omega_j, w_j)$ with $1 \leq j \leq k$, then $f^{(j)} \in L^1_{loc}(\Omega_j)$ and $f^{(j-1)} \in AC_{loc}(\Omega_j)$.

Hypothesis. From now on we assume that w_j is identically 0 in every point of the complement of Ω_j.

We need this hypothesis in order to have complete Sobolev spaces (see [KO] and [RARP1]).

Remark. This hypothesis is satisfied, for example, if we can modify w_j in a set of zero Lebesgue measure in such a way that there exists a sequence $\alpha_n \searrow 0$ with $w_j^{-1}\{(\alpha_n, \infty]\}$ open for every n. If w_j is lower semi-continuous, then it satisfies this condition.

The following definitions also depend on w, although w does not appear explicitly.

Let us consider $w = (w_0, ..., w_k)$ a vectorial weight and $y \in \Delta$. To obtain a greater regularity of the functions in a Sobolev space we construct a modification of the weight w in a neighbourhood of y, using the following version (see a proof in [RARP1, Lemma 3.2]) of the Muckenhoupt inequality (see [Mu], [M, p. 44]). This modified weight is equivalent in some sense to the original one (see Theorem A below).

Muckenhoupt inequality I. Let us consider w_0, w_1 weights in (a, b). Then there exists a positive constant c such that

$$\left\| \int_x^b g(t)\, dt \right\|_{L^\infty([a, b], w_0)} \leq c\, \|g\|_{L^\infty([a, b], w_1)}$$

for any measurable function g in $[a, b]$, if and only if

$$\operatorname*{ess\,sup}_{a<r<b} w_0(r) \int_r^b w_1^{-1} < \infty.$$

DEFINITION 3.5. A vectorial weight $\bar{w} = (\bar{w}_0, ..., \bar{w}_k)$ is a right completion of a vectorial weight w with respect to y if $\bar{w}_k := w_k$ and there is an $\varepsilon > 0$ such that $\bar{w}_j := w_j$ in the complement of $[y, y+\varepsilon]$ and

$$\bar{w}_j := w_j + \tilde{w}_j, \qquad \text{in} \quad [y, y+\varepsilon] \quad \text{for} \quad 0 \leq j < k,$$

where $\tilde{w}_j$ is any weight satisfying:

(i) $\tilde{w}_j \in L^\infty([y, y+\varepsilon])$,

(ii) $\Lambda_\infty(\tilde{w}_j, \bar{w}_{j+1}) < \infty$, with

$$\Lambda_\infty(u, v) := \operatorname*{ess\,sup}_{y<r<y+\varepsilon} u(r) \int_r^{y+\varepsilon} v^{-1}.$$

Muckenhoupt inequality I guarantees that if $f^{(j)} \in L^\infty(w_j)$ and $f^{(j+1)} \in L^\infty(w_{j+1})$, then $f^{(j)} \in L^\infty(\bar{w}_j)$.

EXAMPLE. It can be shown that the following construction is always a completion: we choose $\tilde{w}_j := 0$ if $\bar{w}_{j+1} \notin B_\infty((y, y+\varepsilon])$; if $\bar{w}_{j+1} \in B_\infty([y, y+\varepsilon])$ we set $\tilde{w}_j(x) := 1$ in $[y, y+\varepsilon]$; and if $\bar{w}_{j+1} \in B_\infty((y, y+\varepsilon]) \setminus B_\infty([y, y+\varepsilon])$ we take $\tilde{w}_j(x) := 1$ for $x \in [y+\varepsilon/2, y+\varepsilon]$, and

$$\tilde{w}_j(x) := \min\left\{1, \left(\int_x^{y+\varepsilon} \bar{w}_{j+1}^{-1}\right)^{-1}\right\},$$

for $x \in (y, y+\varepsilon/2)$.

Remarks.

1. We can define a left completion of w with respect to y in a symmetric way.

2. If $\bar{w}_{j+1} \in B_\infty([y, y+\varepsilon])$, then $\Lambda_\infty(\tilde{w}_j, \bar{w}_{j+1}) < \infty$ for any weight $\tilde{w}_j \in L^\infty([y, y+\varepsilon])$. In particular, $\Lambda_\infty(1, \bar{w}_{j+1}) < \infty$.

3. If w, v are two vectorial weights such that $w_j \geqslant c v_j$ for $0 \leqslant j \leqslant k$ and $\bar{v}$ is a right completion of v, then there is a right completion $\bar{w}$ of w, with $\bar{w}_j \geqslant c \bar{v}_j$ for $0 \leqslant j \leqslant k$ (it is enough to take $\tilde{w}_j = \tilde{v}_j$). Also, if w, v are comparable measures, $\bar{v}$ is a right completion of v if and only if it is comparable to a right completion $\bar{w}$ of w.

4. We always have $\bar{w}_k = w_k$ and $\bar{w}_j \geqslant w_j$ for $0 \leqslant j < k$.

DEFINITION 3.6. If w is a vectorial weight, we say that a point $y \in \mathbf{R}$ is right *j*-regular (respectively, left *j*-regular), if there exist $\varepsilon > 0$, a right completion $\bar{w}$ (respectively, left completion) of w, and $j < i \leqslant k$ such that $\bar{w}_i \in B_\infty([y, y+\varepsilon])$ (respectively, $B_\infty([y-\varepsilon, y])$). Also, we say that a point $y \in \mathbf{R}$ is *j*-regular if it is right and left *j*-regular.

Remarks.

1. A point $y \in \mathbf{R}$ is right *j*-regular (respectively, left *j*-regular), if at least one of the following properties is verified:

(a) There exist $\varepsilon > 0$ and $j < i \leqslant k$ such that $w_i \in B_\infty([y, y+\varepsilon])$ (respectively, $B_\infty([y-\varepsilon, y])$). Here we have chosen $\tilde{w}_j = 0$.

(b) There exist $\varepsilon > 0$, $j < i \leqslant k$, $\alpha > 0$, and $\delta < i - j - 1$ such that

$$w_i(x) \geqslant \alpha\, |x-y|^\delta, \qquad \text{for almost every} \quad x \in [y, y+\varepsilon]$$

(respectively, $[y-\varepsilon, y]$). See Lemma 3.4 in [RARP1].

2. If y is right j-regular (respectively, left), then it is also right i-regular (respectively, left) for each $0 \leqslant i \leqslant j$.

3. We can take $i = j+1$ in this definition since by the second remark to Definition 3.5 we can choose $\bar{w}_l = w_l + 1 \in B_\infty([y, y+\varepsilon])$ for $j < l < i$, if $j+1 < i$.

4. If y is not singular for w_j, then $y \in \Omega_j$ and y is $(j-1)$-regular.

When we use this definition we think of a point $\{b\}$ as the union of two half-points $\{b^+\}$ and $\{b^-\}$. With this convention, each one of the following sets

$$(a, b) \cup (b, c) \cup \{b^+\} = (a, b) \cup [b^+, c) \neq (a, c),$$

$$(a, b) \cup (b, c) \cup \{b^-\} = (a, b^-] \cup (b, c) \neq (a, c),$$

has two connected components, and the set

$$(a, b) \cup (b, c) \cup \{b^-\} \cup \{b^+\} = (a, b) \cup (b, c) \cup \{b\} = (a, c)$$

is connected.

We only use this convention in order to study the sets of continuity of functions: we want that if $f \in C(A)$ and $f \in C(B)$, where A and B are union of intervals, then $f \in C(A \cup B)$. With the usual definition of continuity in an interval, if $f \in C([a, b)) \cap C([b, c])$ then we do not have $f \in C([a, c])$. Of course, we have $f \in C([a, c])$ if and only if $f \in C([a, b^-]) \cap C([b^+, c])$, where by definition, $C([b^+, c]) = C([b, c])$ and $C([a, b^-]) = C([a, b])$. This idea can be formalized with a suitable topological space.

Let us introduce some notation. We denote by $\Omega^{(j)}$ the set of j-regular points or half-points, i.e., $y \in \Omega^{(j)}$ if and only if y is j-regular, we say that $y^+ \in \Omega^{(j)}$ if and only if y is right j-regular, and we say that $y^- \in \Omega^{(j)}$ if and only if y is left j-regular. Obviously, $\Omega^{(k)} = \varnothing$ and $\Omega_{j+1} \cup \cdots \cup \Omega_k \subseteq \Omega^{(j)}$.

Remark. If $0 \leqslant j < k$ and I is an interval, $I \subseteq \Omega^{(j)}$, then the set $I \setminus (\Omega_{j+1} \cup \cdots \cup \Omega_k)$ is discrete (see the remark before Definition 7 in [RARP1]).

DEFINITION 3.7. We say that a function h belongs to the class $AC_{loc}(\Omega^{(j)})$ if $h \in AC_{loc}(I)$ for every connected component I of $\Omega^{(j)}$.

DEFINITION 3.8 (Sobolev space). If $w = (w_0, ..., w_k)$ is a vectorial weight, we define the Sobolev space $W^{k,\infty}(\Delta, w)$ as the space of equivalence classes of

$$V^{k,\infty}(\Delta, w) := \{f : \Delta \to \mathbf{R} / f^{(j)} \in AC_{loc}(\Omega^{(j)}) \text{ for } 0 \leqslant j < k \text{ and } \|f^{(j)}\|_{L^\infty(\Delta, w_j)} < \infty \text{ for } 0 \leqslant j \leqslant k\}$$

with respect to the seminorm

$$\|f\|_{W^{k,\infty}(\Delta, w)} := \sum_{j=0}^{k} \|f^{(j)}\|_{L^\infty(\Delta, w_j)}.$$

Remark. If we are interested in functions f with complex values, we only need to apply the results in this paper to the real and imaginary parts of f.

DEFINITION 3.9. If w is a vectorial weight, let us define the space $\mathscr{K}(\Delta, w)$ as

$$\mathscr{K}(\Delta, w) := \{g : \Omega^{(0)} \to \mathbf{R} / g \in V^{k,\infty}(\overline{\Omega^{(0)}}, w),\ \|g\|_{W^{k,\infty}(\overline{\Omega^{(0)}}, w)} = 0\}.$$

$\mathscr{K}(\Delta, w)$ is the equivalence class of 0 in $W^{k,\infty}(\overline{\Omega^{(0)}}, w)$. This concept and its analogue for $1 \leqslant p < \infty$ play an important role in the general theory of Sobolev spaces and in the study of the multiplication operator in Sobolev spaces in particular (see [RARP1], [RARP2], [R1], [R2], and Theorems A and B below).

DEFINITION 3.10. If w is a vectorial weight, we say that (Δ, w) belongs to the class $\mathscr{C}_0$ if there exist compact sets M_n, which are a finite union of compact intervals, such that

(i) M_n intersects at most a finite number of connected components of $\Omega_1 \cup \cdots \cup \Omega_k$,

(ii) $\mathscr{K}(M_n, w) = \{0\}$,

(iii) $M_n \subseteq M_{n+1}$,

(iv) $\bigcup_n M_n = \Omega^{(0)}$.

Remarks.

1. Condition $(\Delta, w) \in \mathscr{C}_0$ is not very restrictive. In fact, the proof of Theorem A below (see [RARP1, Theorem 4.3]) gives that if $\Omega^{(0)} \setminus (\Omega_1 \cup \cdots \cup \Omega_k)$ has only a finite number of points in each connected component of $\Omega^{(0)}$, and $\mathscr{K}(\Delta, w) = \{0\}$, then $(\Delta, w) \in \mathscr{C}_0$.

2. The proof of Theorem A below gives that if for every connected component Λ of $\Omega_1 \cup \cdots \cup \Omega_k$ we have $\mathscr{K}(\overline{\Lambda}, w) = \{0\}$, then $(\Delta, w) \in \mathscr{C}_0$. Condition $\int_\Lambda w_0 > 0$ implies $\mathscr{K}(\overline{\Lambda}, w) = \{0\}$.

3. Since the restriction of a function of $\mathscr{K}(\Delta, w)$ to M_n is in $\mathscr{K}(M_n, w)$ for every n, we have that $(\Delta, w) \in \mathscr{C}_0$ implies $\mathscr{K}(\Delta, w) = \{0\}$.

4. TECHNICAL RESULTS

In this section we collect the theorems we need in order to prove the results in Sections 5 and 6.

The next results, proved in [RARP1], [RARP2], and [R1], play a central role in the theory of Sobolev spaces with respect to measures (see the proofs in [RARP1, Theorems 4.3 and 5.1]). We present here a weak version of these theorems which are enough for our purposes.

THEOREM A. *Let $w=(w_0, ..., w_k)$ be a vectorial weight. Let K_j be a finite union of compact intervals contained in $\Omega^{(j)}$, for $0 \leqslant j < k$, and $\bar{w}$ a right (or left) completion of w. If $(\Delta, w) \in \mathscr{C}_0$, then there exist positive constants $c_1 = c_1(K_0, ..., K_{k-1})$ and $c_2 = c_2(\bar{w}, K_0, ..., K_{k-1})$ such that*

$$c_1 \sum_{j=0}^{k-1} \|g^{(j)}\|_{L^\infty(K_j)} \leqslant \|g\|_{W^{k,\infty}(\Delta, w)},$$

$$c_2 \|g\|_{W^{k,\infty}(\Delta, \bar{w})} \leqslant \|g\|_{W^{k,\infty}(\Delta, w)}, \qquad \forall g \in V^{k,\infty}(\Delta, w).$$

THEOREM B. *Let us consider a vectorial weight $w=(w_0, ..., w_k)$. Assume that we have either* (i) *$(\Delta, w) \in \mathscr{C}_0$ or* (ii) *$\Omega_1 \cup \cdots \cup \Omega_k$ has only a finite number of connected components. Then the Sobolev space $W^{k,\infty}(\Delta, w)$ is complete.*

Remark. See Theorem 5.1 in [RARP1] for further results on completeness.

Lemma 3.3 in [RARP1] gives the following result.

PROPOSITION A. *Let $w=(w_0, ..., w_k)$ be a vectorial weight in $[a, b]$, with $w_{k_0} \in B_\infty((a, b\,])$ for some $0 < k_0 \leqslant k$. If we construct a right completion $\bar{w}$ of w with respect to the point a taking $\varepsilon = b - a$, and $\bar{w}_j = w_j$ for $k_0 \leqslant j \leqslant k$, then there exist positive constants c_j such that*

$$c_j \|g^{(j)}\|_{L^\infty([a,b], \bar{w}_j)} \leqslant \sum_{i=j}^{k_0} \|g^{(i)}\|_{L^\infty([a,b], w_i)} + \sum_{i=j}^{k_0-1} |g^{(i)}(b)|,$$

for all $0 \leqslant j < k_0$ and $g \in V^{k,\infty}([a,b], w)$. In particular, there is a positive constant c such that

$$c \|g\|_{W^{k,\infty}([a,b], \bar{w})} \leqslant \|g\|_{W^{k,\infty}([a,b], w)} + \sum_{j=0}^{k_0-1} |g^{(j)}(b)|,$$

for all $g \in V^{k,\infty}([a,b], w)$.

The following is a particular case of Corollary 4.3 in [RARP1].

COROLLARY A. *Let us consider a vectorial weight* $w=(w_0, ..., w_k)$. *Let* K_j *be a finite union of compact intervals contained in* $\Omega^{(j)}$, *for* $0 \leqslant j < k$. *If* $(\Delta, w) \in \mathscr{C}_0$, *then there exists a positive constant* $c_1 = c_1(K_0, ..., K_{k-1})$ *such that*

$$c_1 \sum_{j=0}^{k-1} \| g^{(j+1)} \|_{L^1(K_j)} \leqslant \| g \|_{W^{k,\infty}(\Delta, w)}, \qquad \forall g \in V^{k,\infty}(\Delta, w).$$

A simple modification in the proof of Corollary A gives Corollary B. Recall that we use $W^{k-m,\infty}(\Delta, w)$ to denote the Sobolev space $W^{k-m,\infty}(\Delta, (w_m, ..., w_k))$.

COROLLARY B. *Let us consider a vectorial weight* $w=(w_0, ..., w_k)$. *For some* $0 < m \leqslant k$, *assume that* $(\Delta, (w_m, ..., w_k)) \in \mathscr{C}_0$. *Let K be a finite union of compact intervals contained in* $\Omega^{(m-1)}$. *Then there exists a positive constant* $c_1 = c_1(K)$ *such that*

$$c_1 \| g \|_{L^1(K)} \leqslant \| g \|_{W^{k-m,\infty}(\Delta, w)}, \qquad \forall g \in V^{k-m,\infty}(\Delta, w).$$

Theorem 3.1 in [RARP2] and its remark give the following result.

THEOREM C. *Let us consider a vectorial weight* $w=(w_0, ..., w_k)$ *with* $(\Delta, w) \in \mathscr{C}_0$. *Assume that K is a finite union of compact intervals* $J_1, ..., J_n$ *and that for every* J_m *there is an integer* $0 \leqslant k_m \leqslant k$ *verifying* $J_m \subseteq \Omega^{(k_m - 1)}$, *if* $k_m > 0$, *and* $\int_{J_m} w_j = 0$ *for* $k_m < j \leqslant k$, *if* $k_m < k$. *If* $w_j \in L^\infty(K)$ *for* $0 < j \leqslant k$, *then there exists a positive constant* c_0 *such that*

$$c_0 \| fg \|_{W^{k,\infty}(\Delta, w)} \leqslant \| f \|_{W^{k,\infty}(\Delta, w)} \Big(\sup_{x \in \Delta} |g(x)| + \| g \|_{W^{k,\infty}(\Delta, w)} \Big),$$

for every $f, g \in V^{k,\infty}(\Delta, w)$ *with* $g' = g'' = \cdots = g^{(k)} = 0$ *in* $\Delta \backslash K$.

Corollary 3.1 in [RARP2] implies the following result.

COROLLARY C. *Let us consider a vectorial weight* $w=(w_0, ..., w_k)$ *in* (a, b) *with* $w_k \in B_\infty([a, b])$, $w \in L^\infty([a, b])$ *and* $\mathscr{K}([a, b], w) = \{0\}$. *Then there exists a positive constant* c_0 *such that*

$$c_0 \| fg \|_{W^{k,\infty}([a,b], w)} \leqslant \| f \|_{W^{k,\infty}([a,b], w)} \| g \|_{W^{k,\infty}([a,b], w)} \qquad \text{for every} \quad f, g \in V^{k,\infty}([a, b], w).$$

The following result is easy to prove.

LEMMA A. *Let us consider $w = (w_0, ..., w_k)$ a vectorial weight with*

$$w_{j+1}(x) \leqslant c_1 |x - a_0| \, w_j(x),$$

for $0 \leqslant j < k$, $a_0 \in \mathbf{R}$ and x in an interval I. Let $\varphi \in C^k(\mathbf{R})$ be such that $\operatorname{supp} \varphi' \subseteq [\lambda, \lambda + t]$, with $\max\{|\lambda - a_0|, |\lambda + t - a_0|\} \leqslant c_2 t$ and $\|\varphi^{(j)}\|_{L^\infty(I)} \leqslant c_3 t^{-j}$ for $0 \leqslant j \leqslant k$. Then, there is a positive constant c_0 which is independent of I, a_0, λ, t, w, φ, and g such that

$$\|\varphi g\|_{W^{k,\infty}(\Delta, w)} \leqslant c_0 \|g\|_{W^{k,\infty}(I, w)},$$

for every $g \in V^{k,\infty}(\Delta, w)$ with $\operatorname{supp}(\varphi g) \subseteq I$.

Remark. The constant c_0 can depend on c_1, c_2, c_3 and k.

The next result is the version for $p = \infty$ of Corollary 3.2 in [R1]. It can be proved as in the case $1 \leqslant p < \infty$.

COROLLARY D. *Let us consider a compact interval I and a vectorial weight $w = (w_0, ..., w_k) \in L^\infty(I)$. Assume that there exist $a_0 \in I$, an integer $0 \leqslant r < k$, and constants $c, \delta > 0$ such that $w_{j+1}(x) \leqslant c\,|x - a_0| \, w_j(x)$ in $[a_0 - \delta, a_0 + \delta] \cap I$, for $r \leqslant j < k$. Then a_0 is neither right nor left r-regular.*

We define now the following functions,

$$\log_1 x = -\log x, \ \log_2 x = \log(\log_1 x), ..., \ \log_n x = \log(\log_{n-1} x).$$

A computation involving Muckenhoupt inequality gives the following result.

PROPOSITION B. *Let us consider a compact interval I and a vectorial weight $w = (w_0, ..., w_k) \in L^\infty(I)$. Assume that there exist $a_0 \in I$, an integer $0 \leqslant r < k$, $n \in \mathbf{N}$, $\delta, c_i > 0$, $\varepsilon_i \geqslant 0$, and $\alpha_i, \gamma_1^i, ..., \gamma_n^i \in \mathbf{R}$ for $r \leqslant i \leqslant k$ such that*

(i) $w_i(x) \asymp e^{-c_i |x - a_0|^{-\varepsilon_i}} |x - a_0|^{\alpha_i} \log_1^{\gamma_1^i} |x - a_0| \cdots \log_n^{\gamma_n^i} |x - a_0|$ *for* $x \in [a_0 - \delta, a_0 + \delta] \cap I$ *and* $r \leqslant i \leqslant k$,

(ii) $\alpha_i \notin \mathbf{N}$ *if* $\varepsilon_i = 0$ *and* $r < i \leqslant k$.

Then there exists a completion $\bar{w}$ of w such that the Sobolev norms $W^{k,p}(I, w)$ and $W^{k,p}(I, \bar{w})$ are comparable and there exists $r \leqslant r_0 \leqslant k$ with $\bar{w}_{j+1}(x) \leqslant c\,|x - a_0| \, \bar{w}_j(x)$ in $[a_0 - \delta, a_0 + \delta] \cap I$, for $r_0 \leqslant j < k$ if $r_0 < k$, and $w_{r_0} \in B_\infty([a_0 - \delta, a_0 + \delta] \cap I)$. In particular, a_0 is $(r_0 - 1)$-regular if $r_0 > 0$.

5. APPROXIMATION IN $W^{K,\infty}(I, W)$

First of all, the next results allow us to deal with weights which can be obtained by "gluing" simpler ones.

THEOREM 5.1. *Let us consider* $-\infty \leqslant a < b < c < d \leqslant \infty$. *Let* $w = (w_0, ..., w_k)$ *be a vectorial weight in* (a, d) *and assume that there exists an interval* $I \subseteq [b, c]$ *with* $w \in L^\infty(I)$ *and* $(I, w) \in \mathscr{C}_0$. *Then* f *can be approximated by functions of* $C^\infty(\mathbf{R})$ *in* $W^{k,\infty}([a, d], w)$ *if and only if it can be approximated by functions of* $C^\infty(\mathbf{R})$ *in* $W^{k,\infty}([a, c], w)$ *and* $W^{k,\infty}([b, d], w)$.

Remark. If $a, d \in \mathbf{R}$ and $w \in L^\infty([a, d])$, the result is also true with P instead of $C^\infty(\mathbf{R})$. This is a consequence of Bernstein's proof of Weierstrass' theorem (see, e.g., [D, p. 113]), which gives a sequence of polynomials converging uniformly up to the kth derivative for any function in $C^k([a, d])$.

Proof. $[\alpha, \beta] \subseteq I$ prove the non-trivial implication. Let us consider $J = [\alpha, \beta] \subset I$ and an integer $0 \leqslant k_1 \leqslant k$, such that $J \subset (b, c)_{k_1} \subseteq (b, c)^{(k_1 - 1)}$ if $k_1 > 0$, and $\int_J w_j = 0$ for $k_1 < j \leqslant k$ if $k_1 < k$.

Let us consider $f \in V^{k,\infty}([a, d], w)$ and $\varphi_1, \varphi_2 \in C^\infty(\mathbf{R})$ such that φ_1 approximates f in $W^{k,\infty}([a, c], w)$ and φ_2 approximates f in $W^{k,\infty}([b, d], w)$.

Set $\theta \in C^\infty(\mathbf{R})$ a fixed function with $0 \leqslant \theta \leqslant 1$, $\theta = 0$ in $(-\infty, \alpha]$ and $\theta = 1$ in $[\beta, \infty)$. It is enough to see that $\theta\varphi_2 + (1-\theta)\,\varphi_1$ approximates f in $W^{k,\infty}([a, d], w)$ or, equivalently, in $W^{k,\infty}(I, w)$. Theorem C with $\Delta = I$ and $K = J$ gives

$$\begin{aligned} &\| f - \theta\varphi_2 - (1-\theta)\,\varphi_1 \|_{W^{k,\infty}(I, w)} \\ &\quad \leqslant \| \theta(f - \varphi_2) \|_{W^{k,\infty}(I, w)} + \| (1-\theta)(f - \varphi_1) \|_{W^{k,\infty}(I, w)} \\ &\quad \leqslant c(\| f - \varphi_2 \|_{W^{k,\infty}(I, w)} + \| f - \varphi_1 \|_{W^{k,\infty}(I, w)}), \end{aligned}$$

and this finishes the proof of the theorem.

THEOREM 5.2. *Let us consider strictly increasing sequences of real numbers* $\{a_n\}$, $\{b_n\}$ *(n belonging to a finite set, to* $\mathbf{Z}$, $\mathbf{Z}^+$, *or* $\mathbf{Z}^-$*) with* $a_{n+1} < b_n$ *for every n. Let* $w = (w_0, ..., w_k)$ *be a vectorial weight in* $(\alpha, \beta) := \bigcup_n (a_n, b_n)$ *with* $-\infty \leqslant \alpha < \beta \leqslant \infty$. *Assume that for each n there exists an interval* $I_n \subseteq [a_{n+1}, b_n]$ *with* $w \in L^\infty(I_n)$ *and* $(I_n, w) \in \mathscr{C}_0$. *Then* f *can be approximated by functions of* $C^\infty(\mathbf{R})$ *in* $W^{k,\infty}([\alpha, \beta], w)$ *if and only if it can be approximated by functions of* $C^\infty(\mathbf{R})$ *in* $W^{k,\infty}([a_n, b_n], w)$ *for each n.*

Proof. We prove the non-trivial implication. Let us consider $\varphi_n \in C^\infty(\mathbf{R})$ which approximates f in $W^{k,\infty}([a_n, b_n], w)$. By the proof of Theorem 5.1 we know that there are $\theta_n \in C^\infty(\mathbf{R})$ and positive constants c_n such that

$$\|f-\theta_n \varphi_{n+1}-(1-\theta_n)\varphi_n\|_{W^{k,\infty}(I_n, w)} \leqslant c_n(\|f-\varphi_n\|_{W^{k,\infty}(I_n, w)}+\|f-\varphi_{n+1}\|_{W^{k,\infty}(I_n, w)}).$$

Now, given $\varepsilon>0$, it is enough to approximate f in $[a_n, b_n]$ with error less than $\varepsilon \min\{1, c_n^{-1}, c_{n-1}^{-1}\}/2$.

THEOREM 5.3. *Let us consider a compact interval I and a vectorial weight $w=(w_0, ..., w_k) \in L^\infty(I)$ such that $w_k \in B_\infty(I)$. Then we have*

$$\begin{aligned} P^{k,\infty}(I, w) = H_3 &:= \{f \in V^{k,\infty}(I, w)/f^{(k)} \in P^{0,\infty}(I, w_k)\} \\ &= H_0 := \{f \in V^{k,\infty}(I, w)/f^{(j)} \in P^{0,\infty}(I, w_j), \text{ for } 0 \leqslant j \leqslant k\} \\ &= \{f: I \to \mathbf{R}/f^{(k-1)} \in AC(I) \text{ and } f^{(k)} \in P^{0,\infty}(I, w_k)\}. \end{aligned}$$

Proof. We prove first $H_3 \subseteq P^{k,\infty}(I, w)$. If $f \in H_3$, let us consider a sequence $\{q_n\}$ of polynomials which converges to $f^{(k)}$ in $L^\infty(I, w_k)$. Let us choose $a \in I$. Then the polynomials

$$Q_n(x) := f(a)+f'(a)(x-a)+\cdots+f^{(k-1)}(a)\frac{(x-a)^{k-1}}{(k-1)!} + \int_a^x q_n(t)\frac{(x-t)^{k-1}}{(k-1)!}\,dt$$

satisfy

$$Q_n^{(j)}(x) = f^{(j)}(a)+\cdots+f^{(k-1)}(a)\frac{(x-a)^{k-j-1}}{(k-j-1)!}+\int_a^x q_n(t)\frac{(x-t)^{k-j-1}}{(k-j-1)!}\,dt,$$

for $0 \leqslant j < k$. Therefore, for $0 \leqslant j < k$,

$$\begin{aligned} |f^{(j)}(x)-Q_n^{(j)}(x)| &= \left|\int_a^x (f^{(k)}(t)-q_n(t))\frac{(x-t)^{k-j-1}}{(k-j-1)!}\,dt\right| \\ &\leqslant c\int_I |f^{(k)}(t)-q_n(t)|\, w_k(t)\, w_k(t)^{-1}\,dt \\ &\leqslant c\,\|f^{(k)}-q_n\|_{L^\infty(I, w_k)}. \end{aligned}$$

Hence, we have for $0 \leqslant j < k$,

$$\| f^{(j)} - Q_n^{(j)} \|_{L^\infty(I, w_j)} \leqslant c \, \| f^{(k)} - q_n \|_{L^\infty(I, w_k)},$$

since $w_j \in L^\infty(I)$. Then we have obtained that $f \in P^{k, \infty}(I, w)$.

Since $\Omega_k = \operatorname{int}(I)$, $\Omega_1 \cup \cdots \cup \Omega_k = \operatorname{int}(I)$ is connected and Theorem B gives that $W^{k, \infty}(I, w)$ is complete; therefore $P^{k, \infty}(I, w) \subseteq H_0$. The content $H_0 \subseteq H_3$ is direct. The last equality is also direct since the fact $w_k \in B_\infty(I)$ gives $\Omega^{(k-1)} = I$. Then $f^{(k-1)} \in AC(I)$ for every $f \in V^{k, \infty}(I, w)$.

THEOREM 5.4. *Let us consider a compact interval I and a vectorial weight* $w = (w_0, ..., w_k) \in L^\infty(I)$, *such that the set S of singular points for* w_k *in I has zero Lebesgue measure. Assume that there exist* $a_0 \in I$, *an integer* $0 \leqslant r < k$, *and constants* $c, \delta > 0$ *such that*

(1) $w_{j+1}(x) \leqslant c \, |x - a_0| \, w_j(x)$ *in* $[a_0 - \delta, a_0 + \delta] \cap I$, *for* $r \leqslant j < k$,

(2) $w_k \in B_\infty(I \setminus \{a_0\})$,

(3) *if* $r > 0$, a_0 *is* $(r-1)$*-regular.*

Then we have

$$\begin{aligned} P^{k, \infty}(I, w) = H_4 := \{ & f \in V^{k, \infty}(I, w) / f^{(k)} \in P^{0, \infty}(I, w_k), \\ & \exists l \in \mathbf{R} \text{ with } \operatorname*{ess\,lim}_{x \in I, x \to a_0} |f^{(r)}(x) - l| \, w_r(x) = 0, \\ & \text{and } \operatorname*{ess\,lim}_{x \in I, x \to a_0} f^{(j)}(x) \, w_j(x) = 0, \text{ for } r < j < k \text{ if } r < k-1 \} \\ = H_0 := \{ & f \in V^{k, \infty}(I, w) / f^{(j)} \in P^{0, \infty}(I, w_j), \ \text{ for } 0 \leqslant j \leqslant k \} \\ = \{ & f : I \to \mathbf{R} / f^{(k-1)} \in AC_{loc}(I \setminus \{a_0\}), \ f^{(k)} \in P^{0, \infty}(I, w_k), \\ & \exists l \in \mathbf{R} \text{ with } \operatorname*{ess\,lim}_{x \in I, x \to a_0} |f^{(r)}(x) - l| \, w_r(x) = 0, \\ & \operatorname*{ess\,lim}_{x \in I, x \to a_0} f^{(j)}(x) \, w_j(x) = 0, \\ & \text{for } r \leqslant j < k \text{ if } r < k-1, \text{ and } f^{(r-1)} \in AC(I) \text{ if } r > 0 \}. \end{aligned}$$

Remark. Muckenhoupt inequality I gives that condition $w_{j+1}(x) \leqslant c_1 |x - a_0| \, w_j(x)$ is not as restrictive as it seems, since many weights can be modified in order to satisfy it (see Proposition B).

Proof. We prove first $H_4 \subseteq P^{k, \infty}(I, w)$. Let us take $f \in H_4$. Without loss of generality we can assume that a_0 is an interior point of I, since the argument is simpler if $a_0 \in \partial I$. Without loss of generality we can assume also $l = 0$, since in other case we can consider $f(x) - l x^r / r!$ instead of $f(x)$

(recall that $w \in L^\infty(I)$). Consider now a function $\varphi \in C_c^\infty(\mathbf{R})$ with $\varphi = 1$ in $[-1, 1]$, $\varphi = 0$ in $\mathbf{R} \setminus (-2, 2)$, and $0 \leqslant \varphi \leqslant 1$ in $\mathbf{R}$. For each $n \in \mathbf{N}$, let us define $\varphi_n(x) := \varphi(n(x - a_0))$ and $h_n := (1 - \varphi_n) f^{(r)}$. We have

$$\|f^{(r)} - h_n\|_{W^{k-r,\infty}(I,w)} = \|\varphi_n f^{(r)}\|_{W^{k-r,\infty}(I,w)}$$

$$\leqslant c_0 \|f^{(r)}\|_{W^{k-r,\infty}([a_0 - 2/n, a_0 + 2/n], w)},$$

since we are in the hypotheses of Lemma A, where $\lambda = a_0 - 2/n$, $t = 4/n$, and we consider the interval $[a_0 - 2/n, a_0 + 2/n]$: observe that $|\lambda - a_0| = |\lambda + t - a_0| = 2/n = t/2$ and

$$\|\varphi_n^{(j)}\|_{L^\infty(\mathbf{R})} = n^j \|\varphi^{(j)}\|_{L^\infty(\mathbf{R})}$$

$$\leqslant 4^k \max\{\|\varphi\|_{L^\infty(\mathbf{R})}, \|\varphi'\|_{L^\infty(\mathbf{R})}, \ldots, \|\varphi^{(k)}\|_{L^\infty(\mathbf{R})}\} \, t^{-j}.$$

Hence, we deduce that $\|f^{(r)} - h_n\|_{W^{k-r,\infty}(I,w)} \to 0$ as $n \to \infty$, since $\operatorname{ess\,lim}_{x \in I, x \to a_0} f^{(j)}(x) \, w_j(x) = 0$, for each $r \leqslant j \leqslant k$ (Lemma 2.2 gives the result for $j = k$ since hypotheses $w_r \in L^\infty(I)$ and (1) give that a_0 is a singularity of type 1 for w_k in I). Therefore, in order to see that $f^{(r)}$ can be approximated by polynomials in $W^{k-r,\infty}(I, w)$ it is enough to see that each h_n can be approximated by polynomials in $W^{k-r,\infty}(I, w)$. Consider weights $w^n := (w_0, \ldots, w_{k-1}, w_{k,n})$ with $w_{k,n} := w_k + \chi_{[a_0 - 1/n, a_0 + 1/n]} \geqslant w_k$. It is direct that $w^n \in L^\infty(I)$ and $w_{k,n} \in B_\infty(I)$. Observe that Corollary 2.1 gives $h_n^{(k-r)} \in P^{0,\infty}(I, w_k)$, since $h_n^{(k-r)} = (1 - \varphi_n) f^{(k)} + F_n$, with $F_n = -\sum_{i=1}^{k-r} \binom{k-r}{i} \varphi_n^{(i)} f^{(k-i)} \in C(I)$ and $1 - \varphi_n \in C(I)$. Hence Theorem 5.3 implies that each h_n can be approximated by polynomials in $W^{k-r,\infty}(I, w^n)$ and consequently in $W^{k-r,\infty}(I, w)$. Therefore, f can be approximated by polynomials in $W^{k-r,\infty}(I, w)$. This finishes the proof if $r = 0$. In other case, hypotheses (2) and (3) give $\Omega^{(r-1)} = I$ and consequently $f^{(r-1)} \in AC(I)$.

Without loss of generality we can assume that there exists $\varepsilon > 0$ such that $[a_0 - \varepsilon, a_0 + \varepsilon]$ is contained in the interior of I and $w_r \geqslant 1$ in $I \setminus [a_0 - \varepsilon, a_0 + \varepsilon]$. In the other case we can change w by w^* with $w_j^* := w_j$ if $j \neq r$ and $w_r^* := w_r + \chi_{I \setminus [a_0 - \varepsilon, a_0 + \varepsilon]}$. It is obvious that it is more complicated to approximate f in $W^{k,\infty}(I, w^*)$ than in $W^{k,\infty}(I, w)$. Therefore, we have $\mathscr{K}([a, b], (w_r, \ldots, w_k)) = \{0\}$ and $([a, b], (w_r, \ldots, w_k)) \in \mathscr{C}_0$ (see Remark 2 to Definition 3.10).

Let us consider a sequence $\{q_n\}$ of polynomials converging to $f^{(r)}$ in $W^{k-r,\infty}(I, w)$. Corollary B gives

$$\|f^{(r)} - q_n\|_{L^1(I)} \leqslant c \, \|f^{(r)} - q_n\|_{W^{k-r,\infty}(I,w)}.$$

The polynomials defined by

$$Q_n(x) := f(a) + f'(a)(x-a) + \cdots + f^{(r-1)}(a)\,\frac{(x-a)^{r-1}}{(r-1)!} + \int_a^x q_n(t)\,\frac{(x-t)^{r-1}}{(r-1)!}\,dt,$$

satisfy

$$\begin{aligned}\|f - Q_n\|_{W^{k,\infty}(I,w)} &\leqslant c\,\|f^{(r)} - q_n\|_{L^1(I)} + \|f^{(r)} - q_n\|_{W^{k-r,\infty}(I,w)}\\ &\leqslant c\,\|f^{(r)} - q_n\|_{W^{k-r,\infty}(I,w)},\end{aligned}$$

and we conclude that the sequence of polynomials $\{Q_n\}$ converges to f in $W^{k,\infty}(I,w)$.

Since $\Omega_k = \operatorname{int}(I)\backslash\{a_0\}$, $\Omega_1 \cup \cdots \cup \Omega_k$ has at most two connected components and Theorem B gives that $W^{k,\infty}(I,w)$ is complete; therefore $P^{k,\infty}(I,w) \subseteq H_0$. Observe that hypotheses $w_r \in L^\infty(I)$ and (1) give that a_0 is a singularity of type 1 for w_j in I, for each $r < j \leqslant k$. By Theorem 2.1 there exists $l \in \mathbf{R}$ with $\operatorname{ess\,lim}_{x\in I,\, x\to a_0} |f^{(r)}(x) - l|\, w_r(x) = 0$, if a_0 is a singularity for w_r in I; in the other case, it is a direct consequence of the continuity of $f^{(r)}$ in a_0. This fact and Lemma 2.2 give $H_0 \subseteq H_4$. The last equality is direct by the definition of $V^{k,\infty}(I,w)$; it is enough to remark that Corollary D and (2) give $\Omega^{(r)} = \Omega^{(r+1)} = \cdots = \Omega^{(k-1)} = I\backslash\{a_0\}$, and (2) and (3) give $\Omega^{(r-1)} = I$ if $r > 0$.

If we apply Theorem 5.1, Theorem 5.4, and Proposition B, we obtain the next result for Jacobi-type weights.

Corollary 5.1. *Consider a vectorial weight w such that $w_j(x) \asymp (x-a)^{\alpha_j}(b-x)^{\beta_j}$ with $\alpha_j, \beta_j \geqslant 0$ for $0 \leqslant j \leqslant k$. Assume that there exist $0 \leqslant r_1$, $r_2 < k$ such that a is r_1-right regular if $r_1 > 0$, b is r_2-left regular if $r_2 > 0$, and verifying either* (i) *$\alpha_{j+1} \geqslant \alpha_j + 1$ for $r_1 \leqslant j < k$ and $\beta_{j+1} \geqslant \beta_j + 1$ for $r_2 \leqslant j < k$,* (ii) *$\alpha_j \in [0,\infty)\backslash\mathbf{Z}^+$ for $r_1 < j \leqslant k$, and $\beta_j \in [0,\infty)\backslash\mathbf{Z}^+$ for $r_2 < j \leqslant k$. Then*

$$P^{k,\infty}([a,b],w) = \{f \in V^{k,\infty}([a,b],w) / f^{(j)} \in P^{0,\infty}([a,b],w_j),\ for\ 0 \leqslant j \leqslant k\}.$$

Remark. The same argument gives a similar result if there are also a finite number of singularities in (a,b), and even if the singularities in $[a,b]$ are of more general type (as in Proposition B).

LEMMA 5.1. *Consider a weight* $w \in B_\infty([a-2\delta, a+2\delta]\backslash\{a\}) \cap L^\infty([a-2\delta, a+2\delta])$ *and a function* $f \in AC_{loc}([a-2\delta, a+2\delta]\backslash\{a\})$, *continuous in a and verifying* $f' \in P^{0,\infty}([a-2\delta, a+2\delta], w)$. *Assume that the set S of singular points of w in* $[a-2\delta, a+2\delta]$ *has zero Lebesgue measure. Then for each* $\varepsilon > 0$ *there exists a function* $g \in AC([a-2\delta, a+2\delta])$ *with* $g' \in P^{0,\infty}([a-2\delta, a+2\delta], w)$, *such that* $g = f$ *in* $[a-2\delta, a-\delta] \cup [a+\delta, a+2\delta]$ *and*

$$\|f-g\|_{L^\infty([a-2\delta, a+2\delta])} + \|f'-g'\|_{L^\infty([a-2\delta, a+2\delta], w)} < \varepsilon .$$

Remark. Similar results are true in the intervals $[a-2\delta, a]$ and $[a, a+2\delta]$.

Proof. Theorem 2.1 gives that there exists $l \in \mathbf{R}$ with $\operatorname{ess\,lim}_{x\to a} |f'(x)-l|\, w(x) = 0$. Without loss of generality we can assume that $l = 0$, since in the other case we can consider $f(x) - lx$ instead of $f(x)$. We construct the function g in the interval $[a-2\delta, a]$. The construction in $[a, a+2\delta]$ is symmetric. If $f' \in L^1([a-2\delta, a])$, we take $g = f$ in $[a-2\delta, a]$. If $f' \notin L^1([a-2\delta, a])$, the facts $f(x) = \int_{a-2\delta}^x f'$ for $x \in [a-2\delta, a)$ and f continuous in a give that $(f')_+, (f')_- \in L^1_{loc}([a-2\delta, a)) \backslash L^1([a-2\delta, a])$.

Assume now that $a \in \operatorname{ess\,cl}\{x \in [a-2\delta, a) \colon f(x) < f(a)\}$. If $a \in \operatorname{ess\,cl}\{x \in [a-2\delta, a) \colon f(x) > f(a)\}$ the argument is symmetric. If $a \notin \operatorname{ess\,cl}\{x \in [a-2\delta, a) \colon f(x) < f(a)\} \cup \operatorname{ess\,cl}\{x \in [a-2\delta, a) \colon f(x) > f(a)\}$ then $f(x) = f(a)$ for $x \in [a-\delta_0, a]$, which contradicts $f' \notin L^1([a-2\delta, a])$.

We claim that $a \in \operatorname{ess\,cl}\{x \in [a-2\delta, a) \colon f(x) < f(a), f'(x) \geqslant 0\}$. If it is not true there exists $\delta_1 > 0$ with $|\{x \in (a-\delta_1, a) \colon f(x) < f(a), f'(x) \geqslant 0\}| = 0$. Consider $x_0 \in (a-\delta_1, a)$ with $f(x_0) < f(a)$. Since f is continuous in x_0, there exists $\delta_2 > 0$ with $f(x) < f(a)$ for $x \in [x_0, x_0+\delta_2)$. Then $f' < 0$ in almost every point in $[x_0, x_0+\delta_2)$, and consequently $f(x) - f(x_0) = \int_{x_0}^x f' < 0$. By this argument it is clear that the set $\{x \in [x_0, a) \colon f(x) \leqslant f(x_0)\}$ is open and closed in $[x_0, a)$; therefore $f(x) \leqslant f(x_0) < f(a)$ for $x \in [x_0, a)$, which contradicts f continuous in a.

Since $|S| = 0$, for each $\varepsilon > 0$ there exists $\alpha \in [a-\delta, a) \backslash S$ with $f(\alpha) < f(a)$, $f'(\alpha) \geqslant 0$, $\|f'\|_{L^\infty([\alpha, a], w)} < \varepsilon/4$ and $|f(x) - f(a)| < \varepsilon/4$ for $x \in [\alpha, a]$. Consider the family of functions $p_{\lambda,\mu}$ in $[\alpha, a]$ defined as follows: for each $\lambda \geqslant 0$ and $0 < \mu < (a-\alpha)/2$, $p_{\lambda,\mu}$ is the function whose graphic is the segment joining $(\alpha, f'(\alpha))$ and $(\alpha+\mu, \lambda)$ in $[\alpha, \alpha+\mu]$, the segment joining $(a-\mu, \lambda)$ and $(a, 0)$ in $[a-\mu, a]$, and is equal to λ in $[\alpha+\mu, a-\mu]$.

It is clear that there exists $\lambda \geqslant 0$ and $0 < \mu < (a-\alpha)/2$ such that the function

$$h_{\lambda,\mu}(x) := \begin{cases} f'(x) & \text{if } x \in [a-2\delta, \alpha] \\ \min((f')_+(x), p_{\lambda,\mu}(x)) & \text{if } x \in (\alpha, a], \end{cases}$$

verifies $f(a)-f(\alpha)=\int_\alpha^a h_{\lambda,\mu}$, since $(f')_+ \in L^1_{loc}([a-2\delta, a))\backslash L^1([a-2\delta, a])$. Observe that $h_{\lambda,\mu}\in L^1([a-2\delta, a])\cap P^{0,\infty}([a-2\delta, a], w)$ (see Corollary 2.1 and Theorem 2.1). For this particular choice of λ and μ, we define $g(x):=f(a)+\int_a^x h_{\lambda,\mu}$ in $[a-2\delta, a]$. We define g in $[a, a+2\delta]$ in a similar way. Conditions $f(a)-f(\alpha)=\int_\alpha^a h_{\lambda,\mu}$ and $h_{\lambda,\mu}=f'$ in $[a-2\delta, \alpha]$ give $g=f$ in $[a-2\delta, \alpha]$. Since $h_{\lambda,\mu}$ does not change its sign in $[\alpha, a]$, we have $|g(x)-g(a)|\leqslant |g(\alpha)-g(a)|=|\int_\alpha^a h_{\lambda,\mu}|=|f(a)-f(\alpha)|<\varepsilon/4$ for every $x\in[\alpha, a]$. Therefore $|g(x)-f(x)|<\varepsilon/2$ for $x\in[\alpha, a]$ and $\|f-g\|_{L^\infty([a-2\delta, a])}<\varepsilon/2$. We also have $|g'(x)|\leqslant |f'(x)|$ in $[\alpha, a]$ and therefore

$$\|f'-g'\|_{L^\infty([a-2\delta, a], w)}\leqslant 2\,\|f'\|_{L^\infty([\alpha, a], w)}<\varepsilon/2.$$

This finishes the proof of the lemma.

THEOREM 5.5. *Let us consider a compact interval $I:=[a, b]$ and a vectorial weight $w=(w_0, ..., w_k)\in L^\infty(I)$. Assume that there exists a finite set $R\subset I$ such that*

(1) *the points of R are singularities for w_k in I,*

(2) $w_k\in B_\infty(I\backslash R)$,

(3) *the points of R are not singular for w_{k-1} in I,*

(4) *the set S of singular points for w_k in I is countable.*

Then we have

$$\begin{aligned} P^{k,\infty}(I, w)=H_5 &:=\{f\in V^{k,\infty}(I, w)/f^{(k)}\in P^{0,\infty}(I, w_k) \\ &\qquad \textit{and } f^{(k-1)} \textit{ is continuous in each point of } R\} \\ &=H_0:=\{f\in V^{k,\infty}(I, w)/f^{(j)}\in P^{0,\infty}(I, w_j), \textit{ for } 0\leqslant j\leqslant k\} \\ &=\{f\colon I\to \mathbf{R}/f^{(k)}\in P^{0,\infty}(I, w_k),\ f^{(k-1)}\in AC_{loc}(I\backslash R), \\ &\qquad \textit{and } f^{(k-1)} \textit{ is continuous in each point of } R\}. \end{aligned}$$

Proof. We prove first $H_5\subseteq P^{k,\infty}(I, w)$. Consider a function $f\in H_5$. Condition (2) gives $f^{(k-1)}\in AC_{loc}(I\backslash R)$. Given $n\in\mathbf{N}$, if we apply a finite number of times Lemma 5.1 (or its remark) to the function $f^{(k-1)}$, we obtain a function $g_n\in AC(I)$ with $g_n'\in P^{0,\infty}(I, w_k)$ and

$$\|f^{(k-1)}-g_n\|_{L^\infty(I)}+\|f^{(k)}-g_n'\|_{L^\infty(I, w_k)}<\frac{1}{n},$$

since $f^{(k-1)}$ is continuous in each point of R, $w_k \in L^\infty(I)$ and $|S|=0$. If $k \geqslant 2$, conditions (2) and (3) give $\Omega^{(k-2)} = I$; hence $f^{(k-2)} \in AC([a, b])$ and the functions

$$G_n(x) := f(a) + f'(a)(x-a) + \cdots + f^{(k-1)}(a)\,\frac{(x-a)^{k-2}}{(k-2)!} + \int_a^x g_n(t)\,\frac{(x-t)^{k-2}}{(k-2)!}\,dt,$$

verify

$$f^{(j)}(x) - G_n^{(j)}(x) = \int_a^x (f^{(k-1)}(t) - g_n(t))\,\frac{(x-t)^{k-j-2}}{(k-j-2)!}\,dt,$$

for $0 \leqslant j \leqslant k-2$, if $k \geqslant 2$. Consequently, since $w \in L^\infty(I)$, we have for any $k \geqslant 1$

$$\|f - G_n\|_{W^{k,\infty}(I,w)} \leqslant c\,\|f^{(k-1)} - g_n\|_{L^\infty(I)} + \|f^{(k)} - g_n'\|_{L^\infty(I,w_k)} \to 0,$$

as $n \to \infty$. For each $n \in \mathbf{N}$, since $g_n' \in L^1(I)$, I is compact and S is countable, by Theorem 2.1 we can approximate g_n' by polynomials with the norm $\|\cdot\|_{L^\infty(I,w_k)} + \|\cdot\|_{L^1(I)}$. An integration argument finishes the proof of $H_5 \subseteq P^{k,\infty}(I, w)$.

Since $\Omega_k = \operatorname{int}(I)\backslash R$, $\Omega_1 \cup \cdots \cup \Omega_k$ has at most a finite number of connected components and Theorem B gives that $W^{k,\infty}(I, w)$ is complete; therefore $P^{k,\infty}(I, w) \subseteq H_0$. Let us take $f \in H_0$. Lemma 2.1 and hypothesis (3) imply that $f^{(k-1)}$ is continuous in each point of R. This gives $H_0 \subseteq H_5$. The last equality is direct by the definition of $V^{k,\infty}(I, w)$, since (2) gives $\Omega^{(k-1)} = I \backslash R$.

THEOREM 5.6. *Let us consider $I := [a, b]$ and a vectorial weight $w = (w_0, \ldots, w_k) \in L^\infty(I)$, with $w_k \in B_\infty((a, b\,])$. Assume that a is a singularity for w_k in I, the set of singularities S for w_k in I has zero Lebesgue measure and $S \cap [a, a+\varepsilon]$ is countable for some $\varepsilon > 0$. If $k \geqslant 2$, assume also that a is right $(k-2)$-regular. Then we have*

$$\begin{aligned} P^{k,\infty}(I, w) = H_6 &:= \{f \in V^{k,\infty}(I, w) / f^{(j)} \in P^{0,\infty}(I, w_j),\ \textit{for}\ j = k-1, k\} \\ &= H_0 := \{f \in V^{k,\infty}(I, w) / f^{(j)} \in P^{0,\infty}(I, w_j),\ \textit{for}\ 0 \leqslant j \leqslant k\} \\ &= \{f: I \to \mathbf{R} / f^{(k-2)} \in AC(I)\ \textit{if}\ k \geqslant 2,\ \ f^{(k-1)} \in AC_{loc}((a, b]) \\ &\qquad \textit{and}\ f^{(j)} \in P^{0,\infty}(I, w_j),\ \textit{for}\ j = k-1, k\}. \end{aligned}$$

Remark. A similar result is true for $w_k \in B_\infty([a, b))$ or $w_k \in B_\infty((a, b))$.

Proof. We prove first $H_6 \subseteq P^{k,\infty}(I, w)$. Fix a function f in H_6. Take a closed interval $J := [\alpha, \beta] \subset (a, a+\varepsilon)$; we have $w_k \in B_\infty(J)$ and therefore $f^{(k-1)} \in AC(J)$. Without loss of generality we can assume that $w_0 \geqslant 1$ in J, since in other case we can consider $w^* := (w_0^*, w_1, ..., w_k)$ with $w_0^* := w_0 + \chi_J$, and it is more difficult to approximate f in $W^{k,\infty}(I, w^*)$ than in $W^{k,\infty}(I, w)$. Remark 2 to Definition 3.10 gives that $(J, w) \in \mathscr{C}_0$. Theorems 5.1 and 5.3 give that it is enough to prove the inclusion in the interval $[a, \beta]$. Therefore, without loss of generality we can assume that the set S of singularities for w_k in $[a, b]$ is countable. By Theorem 2.1, there exist $l_j \in \mathbf{R}$ such that $\operatorname{ess\,lim}_{x \in I, x \to a} |f^{(j)}(x) - l_j| \, w_j(x) = 0$, for $j = k-1, k$ (if a is not singular for w_{k-1} in I, this fact is direct for $k-1$ with $l_{k-1} = f^{(k-1)}(a)$). Without loss of generality we can assume that $l_{k-1} = l_k = 0$, i.e.,

$$\operatorname*{ess\,lim}_{x \in I,\, x \to a} |f^{(j)}(x)| \, w_j(x) = 0, \tag{5.1}$$

for $j = k-1, k$, since in the other case we can consider $f(x) - l_{k-1}(x-a)^{k-1}/(k-1)! - l_k(x-a)^k/k!$ instead of $f(x)$. Observe that $w_k \in B_\infty((a, b\,])$ gives $f^{(k-1)} \in AC_{loc}((a, b\,])$.

Let us choose $0 < t_n \leqslant 1/n$ such that $a + t_n \notin S$ and

$$|f^{(k-1)}(a+t_n)| \leqslant \inf_{x \in (a,\, a+1/n]} |f^{(k-1)}(x)| + \frac{1}{n}. \tag{5.2}$$

Choose functions g_n verifying $g_n = f^{(k)}$ in $[a+t_n, b]$, $g_n \in C([a, a+t_n])$, $|g_n| \leqslant |f^{(k)}|$ in $[a, a+t_n]$, and $\int_a^{a+t_n} |g_n| < 1/n$ (recall that $f^{(k)}$ is continuous in a neighbourhood of $a+t_n$ by Lemma 2.1). Since $|S| = 0$, Theorem 2.1 gives $g_n \in P^{0,\infty}(I, w_k)$. Observe that $g_n \in L^1(I)$, since

$$\begin{aligned}\|g_n\|_{L^1(I)} &< \frac{1}{n} + \int_{a+t_n}^b |f^{(k)}| \, w_k \, w_k^{-1} \\ &\leqslant \frac{1}{n} + \|f^{(k)}\|_{L^\infty(I, w_k)} \|w_k^{-1}\|_{L^1([a+t_n, b])} < \infty.\end{aligned}$$

Define

$$f_n(x) := f(b) + \cdots + f^{(k-1)}(b) \frac{(x-b)^{k-1}}{(k-1)!} + \int_b^x g_n(t) \frac{(x-t)^{k-1}}{(k-1)!} \, dt.$$

Conditions $\int_a^{a+t_n} |g_n| < 1/n$ and (5.2) give

$$|f_n^{(k-1)}(x)| \leqslant |f^{(k-1)}(x)| + \frac{2}{n}, \tag{5.3}$$

for $x \in (a, a+t_n]$. By (5.1) we have

$$\|f^{(k)} - f_n^{(k)}\|_{L^\infty(I, w_k)} = \|f^{(k)} - g_n\|_{L^\infty(I, w_k)} \leqslant 2 \, \|f^{(k)}\|_{L^\infty([a, a+t_n], w_k)} \to 0 \, ,$$

as $n \to \infty$. By (5.1) and (5.3), we also have as $n \to \infty$

$$\begin{aligned} \|f^{(k-1)} - f_n^{(k-1)}\|_{L^\infty(I, w_{k-1})} & \\ \leqslant \|f^{(k-1)}\|_{L^\infty([a, a+t_n], w_{k-1})} &+ \|f_n^{(k-1)}\|_{L^\infty([a, a+t_n], w_{k-1})} \\ \leqslant 2 \, \|f^{(k-1)}\|_{L^\infty([a, a+t_n], w_{k-1})} &+ \frac{2}{n} \, \|w_{k-1}\|_{L^\infty([a, b])} \to 0. \end{aligned}$$

These facts give that $\lim_{n \to \infty} \|f^{(k-1)} - f_n^{(k-1)}\|_{W^{1, \infty}(I, w)} = 0$. Assume now $k \geqslant 2$. Choose a compact interval $J_0 \subset (a, b) = \Omega_k$; we have $f^{(k-1)} \in AC(J_0)$ and then f belongs to $V^{k, \infty}([a, b], \tilde{w})$ with $\tilde{w} = (w_0, ..., w_{k-2}, \tilde{w}_{k-1}, w_k)$ and $\tilde{w}_{k-1} = w_{k-1} + \chi_{J_0}$. Observe that $\mathscr{K}(I, (\tilde{w}_{k-1}, w_k)) = \{0\}$ and even $(I, (\tilde{w}_{k-1}, w_k)) \in \mathscr{C}_0$, since $\Omega_k = (a, b)$ (see Remark 2 to Definition 3.10). It is obvious that it is more complicated to approximate f in $W^{k, \infty}(I, \tilde{w})$ than in $W^{k, \infty}(I, w)$. Therefore, without loss of generality we can assume that $(I, (w_{k-1}, w_k)) \in \mathscr{C}_0$.

Since $\Omega^{(k-1)} = (a, b\,]$ and a is right $(k-2)$-regular, we have $\Omega^{(k-2)} = I$, and hence Corollary B gives

$$\|f^{(k-1)} - f_n^{(k-1)}\|_{L^1(I)} \leqslant c \, \|f^{(k-1)} - f_n^{(k-1)}\|_{W^{1, \infty}(I, (w_{k-1}, w_k))} \, .$$

It is clear that

$$f_n(x) = f(b) + \cdots + f^{(k-2)}(b) \, \frac{(x-b)^{k-2}}{(k-2)!} + \int_b^x f_n^{(k-1)}(t) \, \frac{(x-t)^{k-2}}{(k-2)!} \, dt,$$

and consequently

$$f^{(j)}(x) - f_n^{(j)}(x) = \int_b^x (f^{(k-1)}(t) - f_n^{(k-1)}(t)) \, \frac{(x-t)^{k-j-2}}{(k-j-2)!} \, dt,$$

for $0 \leqslant j \leqslant k-2$, if $k \geqslant 2$. Hence we have that

$$\begin{aligned} \|f^{(j)} - f_n^{(j)}\|_{L^\infty(I, w_j)} &\leqslant c \, \|f^{(k-1)} - f_n^{(k-1)}\|_{L^1(I)} \\ &\leqslant c \, \|f^{(k-1)} - f_n^{(k-1)}\|_{W^{1, \infty}(I, (w_{k-1}, w_k))} \, , \end{aligned}$$

for $0 \leqslant j \leqslant k-2$ and we conclude that $\{f_n\}$ converges to f in $W^{k, \infty}(I, w)$. Therefore, for any $k \geqslant 1$, in order to finish the proof of this inclusion it is enough to find $Q_n \in P$ with $\lim_{n \to \infty} \|f_n - Q_n\|_{W^{k, \infty}(I, w)} = 0$. Since S is countable and $g_n \in P^{0,\infty}(I, w_k) \cap L^1(I)$, Theorem 2.1 gives that there exists $h_n \in P$ with $\|g_n - h_n\|_{L^\infty(I, w_k)} + \|g_n - h_n\|_{L^1(I)} < 1/n$. Hence the polynomials

$$Q_n(x) := f(b) + \cdots + f^{(k-1)}(b) \frac{(x-b)^{k-1}}{(k-1)!} + \int_b^x h_n(t) \frac{(x-t)^{k-1}}{(k-1)!} \, dt$$

satisfy the inequality $c \|f_n - Q_n\|_{W^{k,\infty}(I, w)} \leqslant \|g_n - h_n\|_{L^1(I)} + \|g_n - h_n\|_{L^\infty(I, w_k)}$, and consequently we obtain $\lim_{n \to \infty} \|f_n - Q_n\|_{W^{k,\infty}(I, w)} = 0$. Therefore $H_6 \subseteq P^{k,\infty}(I, w)$.

Since $\Omega_k = \operatorname{int}(I)$, $\Omega_1 \cup \cdots \cup \Omega_k = \operatorname{int}(I)$ is connected and Theorem B gives that $W^{k,\infty}(I, w)$ is complete; therefore $P^{k,\infty}(I, w) \subseteq H_0$. The content $H_0 \subseteq H_6$ is direct. The last equality is direct by the definition of $V^{k,\infty}(I, w)$, since $\Omega^{(k-1)} = (a, b\,]$, and $\Omega^{(k-2)} = [a, b]$ if $k \geqslant 2$.

THEOREM 5.7. *Let us consider $I := [a, b]$ and a vectorial weight $w = (w_0, ..., w_k) \in L^\infty(I)$, with $w_k \in B_\infty((a, b\,])$. Assume that a is a singularity for w_k in I, the set S of singularities for w_k in I has zero Lebesgue measure and $S \cap [a, a+\varepsilon]$ is countable for some $\varepsilon > 0$. If $k \geqslant 2$, assume also that $w|_{[a, a+\varepsilon]}$ is a right completion of* $(0, ..., 0, w_{k-1}, w_k)$. *Then we have*

$$P^{k,\infty}(I, w) = \{ f \in V^{k,\infty}(I, w) / f^{(j)} \in P^{0,\infty}(I, w_j), \text{ for } j = k-1, k \}.$$

Remark. A similar result is true for $w_k \in B_\infty([a, b))$ or $w_k \in B_\infty((a, b))$.

Proof. If $k = 1$, the result is a direct consequence of Theorem 5.6. Assume that $k \geqslant 2$. The argument follows the same lines as the one in the proof of Theorem 5.6. By Theorems 5.1 and 5.3 we can assume that $b = a + \varepsilon$. Given a function f with $f^{(j)} \in P^{0,\infty}(I, w_j)$, for $j = k-1, k$, let us consider the sequence $\{f_n\}$ in the proof of Theorem 5.6. As in the proof of Theorem 5.6, we also have $f_n^{(k-1)} \to f^{(k-1)}$ in $W^{1,\infty}(I, w)$, as $n \to \infty$.

By Proposition A there is a positive constant c such that

$$c \|g\|_{W^{k,\infty}(I, w)} \leqslant \|g\|_{W^{k,\infty}(I, (0, ..., 0, w_{k-1}, w_k))} + \sum_{j=0}^{k-1} |g^{(j)}(b)|,$$

for all $g \in V^{k,\infty}(I, w)$. Since $(f - f_n)^{(j)}(b) = 0$ for $0 \leqslant j < k$, we have

$$\|f - f_n\|_{W^{k,\infty}(I, w)} \leqslant c \, \|f^{(k-1)} - f_n^{(k-1)}\|_{W^{1,\infty}(I, w)},$$

and we conclude that $\{f_n\}$ converges to f in $W^{k,\infty}(I, w)$. The proof finishes with the arguments in the proof of Theorem 5.6.

6. NON-BOUNDED INTERVALS

Although the main interest in this section is the case of non-bounded intervals, the following result can be applied to the case of compact intervals.

THEOREM 6.1. *Let us consider a vectorial weight $w=(w_0, ..., w_k)$. Assume that there exist $a \in \Delta$ and a positive constant c such that*

$$c \, \|g\|_{W^{k,\infty}(\Delta, w)} \leqslant |g(a)| + |g'(a)| + \cdots + |g^{(k-1)}(a)| + \|g^{(k)}\|_{L^\infty(\Delta, w_k)}, \tag{6.1}$$

for every $g \in V^{k,\infty}(\Delta, w)$. Then,

$$P^{k,\infty}(\Delta, w) = \{f\colon \Delta \to \mathbf{R} / f^{(k)} \in P^{0,\infty}(\Delta, w_k)\}.$$

Proof. We prove the non-trivial inclusion. Let us consider a fixed function f with $f^{(k)} \in P^{0,\infty}(\Delta, w_k)$. Choose a sequence $\{q_n\}$ of polynomials which converges to $f^{(k)}$ in $L^\infty(\Delta, w_k)$. Then the polynomials

$$Q_n(x) := f(a) + f'(a)(x-a) + \cdots + f^{(k-1)}(a) \frac{(x-a)^{k-1}}{(k-1)!} + \int_a^x q_n(t) \frac{(x-t)^{k-1}}{(k-1)!} \, dt$$

satisfy

$$c \, \|f - Q_n\|_{W^{k,\infty}(\Delta, w)} \leqslant \|f^{(k)} - Q_n^{(k)}\|_{L^\infty(\Delta, w_k)} = \|f^{(k)} - q_n\|_{L^\infty(\Delta, w_k)},$$

since $(f - Q_n)^{(j)}(a) = 0$ for $0 \leqslant j < k$, and we conclude that the sequence of polynomials $\{Q_n\}$ converges to f in $W^{k,\infty}(\Delta, w)$.

We show now that Theorem 6.1 is very useful finding a wide class of measures satisfying (6.1). The following inequality is similar to the Muckenhoupt inequality which can be found in [Mu] and [M, p. 40].

PROPOSITION (Muckenhoupt inequality II). *Let us consider two weights w_0, w_1 in $(0, \infty)$. Then there exists a positive constant c such that*

$$\left\| \int_0^x g(t) \, dt \right\|_{L^\infty([0,\infty), w_0)} \leqslant c \, \|g\|_{L^\infty([0,\infty), w_1)} \tag{6.2}$$

for any measurable function g in $(0, \infty)$, *if and only if*

$$B := \operatorname*{ess\,sup}_{r>0} w_0(r) \int_0^r w_1(t)^{-1}\, dt < \infty.$$

Furthermore, the best constant c in (6.2) *is B.*

Remark. A similar result is true for the intervals (a, ∞) and $(-\infty, a)$, with $a \in \mathbf{R}$.

Proof. Assume that $B < \infty$. We have

$$\left| \int_0^r g(t)\, dt \right| w_0(r) \leqslant \int_0^r |g(t)|\, w_1(t)\, w_1(t)^{-1}\, dt\, w_0(r)$$

$$\leqslant \|g\|_{L^\infty([0,r],\, w_1)}\, w_0(r) \int_0^r w_1(t)^{-1}\, dt,$$

and this implies (6.2) with $c = B$. If (6.2) holds, the choice of the function $g := w_1^{-1}$ gives $B \leqslant c < \infty$.

LEMMA 6.1. *Assume that* $w_0(x) \leqslant c_0 x^{\alpha_0} e^{-\lambda x^\varepsilon}$ *and* $w_1(x) \geqslant c_1 x^{\alpha_1} e^{-\lambda x^\varepsilon}$, *for* $x \geqslant A$, $w_0 \in L^\infty([0, A])$, $w_1 \in B_\infty([0, A])$, *with* $\lambda, \varepsilon, c_0, c_1, A > 0$ *and* $\alpha_0, \alpha_1 \in \mathbf{R}$. *If* $\alpha_0 \leqslant \alpha_1 + \varepsilon - 1$, *then* w_0, w_1 *satisfy Muckenhoupt inequality II.*

Proof. First of all observe that

$$(x^a e^{bx^\varepsilon})' = x^{a-1} e^{bx^\varepsilon} (a + b\varepsilon x^\varepsilon)\,.$$

This implies $(x^a e^{bx^\varepsilon})' \asymp x^{a+\varepsilon-1} e^{bx^\varepsilon}$, as $x \to \infty$, if $b > 0$. Therefore

$$\int_A^r x^a e^{bx^\varepsilon}\, dx \asymp r^{a+1-\varepsilon} e^{br^\varepsilon},$$

as $r \to \infty$. Hence, we have as $r \to \infty$

$$\int_0^r w_1(x)^{-1}\, dx \asymp \int_A^r w_1(x)^{-1}\, dx \leqslant c \int_A^r x^{-\alpha_1} e^{\lambda x^\varepsilon}\, dx \asymp r^{-\alpha_1+1-\varepsilon} e^{\lambda r^\varepsilon}.$$

The expression $w_0(r) \int_0^r w_1^{-1}$ is bounded for r in a compact set; it is bounded for big r, if

$$\lim_{r\to\infty} r^{\alpha_0} e^{-\lambda r^\varepsilon} r^{-\alpha_1+1-\varepsilon} e^{\lambda r^\varepsilon} < \infty\,.$$

This condition holds since $\alpha_0 \leqslant \alpha_1 + \varepsilon - 1$.

Lemma 6.2. *Assume that* $w_0(x) \leqslant k_0 x^{\beta_0}$ *and* $w_1(x) \geqslant k_1 x^{\beta_1}$, *for* $0 < x < b$, *with* $k_0, k_1 > 0$, $\beta_0 > 0$ *and* $\beta_1 \in \mathbf{R}$. *If* $\beta_0 \geqslant \beta_1 - 1$, *then* w_0, w_1 *satisfy Muckenhoupt inequality I, with* $a = 0$.

Proof. If $\beta_1 > 1$, we have

$$\int_r^b w_1(x)^{-1}\, dx \leqslant c \int_r^b x^{-\beta_1}\, dx \asymp r^{1-\beta_1}.$$

If $\beta_1 > 1$, the expression $F(r) := w_0(r) \int_r^b w_1^{-1}$ is bounded for $r \in [\varepsilon, b]$ (with $\varepsilon > 0$); it is bounded for $r \in (0, \varepsilon)$, if

$$\lim_{r \to 0^+} r^{\beta_0} r^{1-\beta_1} < \infty\ .$$

This condition holds since $\beta_0 \geqslant \beta_1 - 1$. If $\beta_1 \leqslant 1$, we obtain similarly that $F(r)$ is bounded since $\beta_0 > 0$ and

$$F(r) \leqslant c r^{\beta_0} \log \frac{1}{r},$$

for small r.

These lemmas give the following results.

Proposition 6.1. *Consider a vectorial weight* w *in* $(0, \infty)$, *with*

(1) $w_j(x) \leqslant c_j x^{\beta_j}$, *for* $0 \leqslant j < k$, $w_k(x) \geqslant c_k x^{\beta_k}$, *in* $(0, a)$,

(2) $w_j(x) \leqslant c_j x^{\alpha + (k-j)(\varepsilon - 1)} e^{-\lambda x^{\varepsilon}}$, *for* $0 \leqslant j < k$, $w_k(x) \geqslant c_k x^{\alpha} e^{-\lambda x^{\varepsilon}}$, *in* (a, ∞),

where $\alpha \in \mathbf{R}$, $a, \varepsilon, \lambda, c_j > 0$ *for* $0 \leqslant j \leqslant k$, *and* $\beta_j > 0$ *for* $0 \leqslant j < k$. *If* $\beta_j \geqslant \beta_k - (k-j)$, *for* $0 \leqslant j < k$, *then*

$$P^{k,\infty}([0,\infty), w) = \{f\colon [0,\infty) \to \mathbf{R} / f^{(k)} \in P^{0,\infty}([0,\infty), w_k)\}.$$

Proof. An induction argument with Lemma 6.1 in (a, ∞) instead of $(0, \infty)$, gives for $0 \leqslant j < k$ and $f \in V^{k,\infty}([a,\infty), w)$,

$$\left\| f^{(j)}(x) - f^{(j)}(a) - \cdots - f^{(k-1)}(a) \frac{(x-a)^{k-j-1}}{(k-j-1)!} \right\|_{L^\infty([a,\infty), w_j)}$$

$$\leqslant c\, \|f^{(k)}\|_{L^\infty([a,\infty), w_k)},$$

and therefore

$$c\,\|f^{(j)}\|_{L^\infty([a,\infty),\,w_j)} \leqslant \|f^{(k)}\|_{L^\infty([a,\infty),\,w_k)} + \sum_{i=j}^{k-1} |f^{(i)}(a)|,$$

for $0 \leqslant j < k$ and $f \in V^{k,\infty}([a,\infty), w)$. Consequently, we have

$$c\,\|f\|_{W^{k,\infty}([a,\infty),\,w)} \leqslant \|f^{(k)}\|_{L^\infty([a,\infty),\,w_k)} + \sum_{j=0}^{k-1} |f^{(j)}(a)|, \tag{6.3}$$

for all $f \in V^{k,\infty}([a,\infty), w)$. If we use now Lemma 6.2 in $(0, a)$, a similar argument gives

$$c\,\|f\|_{W^{k,\infty}([0,a],\,w)} \leqslant \|f^{(k)}\|_{L^\infty([0,a],\,w_k)} + \sum_{j=0}^{k-1} |f^{(j)}(a)|, \tag{6.4}$$

for all $f \in V^{k,\infty}([0,a], w)$. Theorem 6.1, (6.3), and (6.4) give the proposition.

PROPOSITION 6.2. *Consider a vectorial weight w in* $\mathbf{R}$, *with*

(1) $w_j(x) \leqslant c_j\,|x|^{\alpha+(k-j)(\varepsilon-1)}\,e^{-\lambda|x|^\varepsilon}$, *for* $0 \leqslant j < k$, $w_k(x) \geqslant c_k\,|x|^\alpha\,e^{-\lambda|x|^\varepsilon}$, *in* (B, ∞),

(2) $w_j(x) \leqslant c_j\,|x|^{\alpha'+(k-j)(\varepsilon'-1)}\,e^{-\lambda'|x|^{\varepsilon'}}$, *for* $0 \leqslant j < k$, $w_k(x) \geqslant c_k\,|x|^{\alpha'}\,e^{-\lambda'|x|^{\varepsilon'}}$, *in* $(-\infty, -A)$,

(3) $w_j(x) \in L^\infty([-A, B])$, *for* $0 \leqslant j < k$, $w_k(x) \in B_\infty([-A, B])$,

where $\alpha, \alpha' \in \mathbf{R}$, $A, B, \varepsilon, \varepsilon', \lambda, \lambda' > 0$ *and* $c_j > 0$, *for* $0 \leqslant j \leqslant k$. *Then*

$$P^{k,\infty}(\mathbf{R}, w) = \{f\colon \mathbf{R} \to \mathbf{R} / f^{(k)} \in P^{0,\infty}(\mathbf{R}, w_k)\}.$$

Remark. The same result is true if we change $\mathbf{R}$ by $(0, \infty)$.

Proof. The argument is similar to the one in Proposition 6.1, with 0 instead of a. In this case, we only use Lemma 6.1.

We can obtain similar results for weights of fast decreasing degree. The following results are not sharp since the sharp results are hard to write and do not involve any new idea.

Define inductively the functions $\exp_{\lambda_1,\ldots,\lambda_n}$ as follows:

$$\exp_\lambda(t) := \exp(\lambda t), \qquad \exp_{\lambda_1,\ldots,\lambda_n}(t) := \exp(\lambda_1 \exp_{\lambda_2,\ldots,\lambda_n}(t)).$$

LEMMA 6.3. *Consider a scalar weight* $w(x) \asymp \exp_{-\lambda_1, \lambda_2, ..., \lambda_n}(x^\varepsilon)$ *in* $(0, \infty)$, *where we have* $n > 1$ *and* $\varepsilon, \lambda_1, \lambda_2, ..., \lambda_n > 0$. *Then* w, w *satisfy Muckenhoupt inequality II.*

Proof. A straightforward computation shows that the derivative of the function

$$x^{1-\varepsilon} \prod_{i=2}^{n} \exp_{-\lambda_i, \lambda_{i+1}, ..., \lambda_n}(x^\varepsilon),$$

converges to zero as $x \to \infty$. Now, if $b > 0$ we have that

$$\frac{d}{dx}\left(\exp_{b, \lambda_2, ..., \lambda_n}(x^\varepsilon)\, x^{1-\varepsilon} \prod_{i=2}^{n} \exp_{-\lambda_i, \lambda_{i+1}, ..., \lambda_n}(x^\varepsilon)\right) \asymp \exp_{b, \lambda_2, ..., \lambda_n}(x^\varepsilon),$$

in $(1, \infty)$. Hence we have that

$$\int_0^r w^{-1} \asymp \exp_{\lambda_1, \lambda_2, ..., \lambda_n}(r^\varepsilon)\, r^{1-\varepsilon} \prod_{i=2}^{n} \exp_{-\lambda_i, \lambda_{i+1}, ..., \lambda_n}(r^\varepsilon),$$

in $(1, \infty)$. Therefore

$$w(r) \int_0^r w^{-1} \asymp r^{1-\varepsilon} \prod_{i=2}^{n} \exp_{-\lambda_i, \lambda_{i+1}, ..., \lambda_n}(r^\varepsilon),$$

in $(1, \infty)$. This finishes the proof, since $w \in L^\infty([0, \infty))$.

PROPOSITION 6.3. *Consider a vectorial weight* w, *with* $w_j(x) \leqslant c_j \exp_{-\lambda_1, \lambda_2, ..., \lambda_n}(|x|^\varepsilon)$ *in* $\mathbf{R}$, *for* $0 \leqslant j < k$, $w_k(x) \geqslant c_k \exp_{-\lambda_1, \lambda_2, ..., \lambda_n}(|x|^\varepsilon)$ *in* $\mathbf{R}$, *where* $n > 1$ *and* $\varepsilon, \lambda_1, \lambda_2, ..., \lambda_n, c_0, c_1, ..., c_k > 0$. *Then*

$$P^{k,\infty}(\mathbf{R}, w) = \{f : \mathbf{R} \to \mathbf{R} / f^{(k)} \in P^{0,\infty}(\mathbf{R}, w_k)\}.$$

Remark. The same result is true if we change $\mathbf{R}$ by $(0, \infty)$.

Proof. It is enough to follow the argument in the proof of Proposition 6.1, using Lemma 6.3 instead of Lemmas 6.1 and 6.2.

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